An exotic Deligne-Langlands correspondence for symplectic groups

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Abstract

Let $G = Sp(2n, \mathbb{C})$ be a complex symplectic group. We introduce a $G \times (\mathbb{C}^{\times})^{\ell+1}$-variety $\mathfrak{N}_{\ell}$, which we call the ℓ-exotic nilpotent cone. Then, we realize the Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$ with three parameters via equivariant algebraic K-theory in terms of the geometry of $\mathfrak{N}_2$. This enables us to establish a Deligne-Langlands type classification of simple $\mathbb{H}$-modules under a mild assumption on parameters. As applications, we present a character formula and multiplicity formulas of $\mathbb{H}$-modules.

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Introduction

In their celebrated paper [KL87], Kazhdan and Lusztig gave a classification of simple modules of an affine Hecke algebra $\mathbb{H}$ with one-parameter in terms of the geometry of nilpotent cones. (It is also done by Ginzburg, c.f. [CG97].) Since some of the affine Hecke algebras admit two or three parameters, it is natural to extend their result to multi-parameter cases. (It is called the unequal parameter case.) Lusztig realized the "graded version" of $\mathbb{H}$ (with unequal parameters) via several geometric means [Lu88, Lu89, Lu95b] (c.f. [Lu03]) and classified their representations in certain cases. Unfortunately, his geometries admit essentially

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only one parameter. As a result, his classification is restricted to the case where all of the parameters are certain integral power of a single parameter. It is enough for his main interest, the study of representations of p-adic groups (c.f. [Lu95a]). However, there are many areas of mathematics which wait for the full-representation theory of Hecke algebras with unequal parameters (see e.g. Macdonald's book [Mc03] and its featured review in MathSciNet).

In this paper, we give a realization of all simple modules of the Hecke algebra of type $C_n^{(1)}$ with three parameters by introducing a variety which we call *the ℓ-exotic nilpotent cone* (c.f. §1.1). Our framework works for all parameters and realizes the whole Hecke algebra (Theorem A) and its specialization to each central character. Unfortunately, the study of our geometry becomes harder for some parameters and the result becomes less explicit in such cases. Even so, our result gives a definitive classification of simple modules of affine Hecke algebras of type $B_n^{(1)}$ and $C_n^{(1)}$ for almost all parameters including so-called real central character case. (See the argument after Theorem D.)

Let G be the complex symplectic group $Sp(2n, \mathbb{C})$. We fix its Borel subgroup B and a maximal torus $T \subset B$. Let R be the root system of (G, T). We embed R into a n-dimensional Euclid space $\oplus_i \mathbb{C}\epsilon_i$ as $R = \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\}$. We define $V_1 := \mathbb{C}^{2n}$ and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. For each non-negative integer ℓ, we put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation*. Let $\mathbb{V}_\ell^+$ be the positive part of $\mathbb{V}_\ell$ (for precise definition, see §1). We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. This is the G-variety which we refer as *the ℓ-exotic nilpotent cone*. We put $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We have a natural G_ℓ-action on F_ℓ (and Z_ℓ). (In fact, the variety F_ℓ admits an action of $G \times GL(\ell, \mathbb{C}) \times \mathbb{C}^\times$. We use only a restricted action in this paper.)

Assume that $\mathbb{H}$ is the Hecke algebra with unequal parameters of type $C_n^{(1)}$ (c.f. Definition 2.1). This algebra has three parameters q_0, q_1, q_2. All affine Hecke algebras of classical type with two parameters are obtained from $\mathbb{H}$ by suitable specializations of parameters (c.f. Remark 2.2).

Theorem A (= Theorem 2.8). *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_\mathbb{Z} K^{G_2}(Z_2)$$

as algebras.

The Ginzburg theory suggests a classification of simple $\mathbb{H}$-modules by the G-conjugacy classes of the following Langlands parameters:

Definition B (Langlands parameters).

1) A triple $\vec{q} := (q_0, q_1, q_2) \in (\mathbb{C}^\times)^3$ is said to be admissible if $q_0 \neq q_1$, q_2 is not a root of unity of order $\leq 2n$, $q_0 q_1^{\pm 1} \neq q_2^m$ for $|m| < n$;

2) A pair $(a, X) = (s, \vec{q}, X_0 \oplus X_1 \oplus X_2) \in G_2 \times \mathfrak{N}_2$ is called an admissible parameter iff s is semisimple, $\vec{q}$ is admissible, and $sX_i = q_i X_i$ for $i = 0, 1, 2$.

For $a = (s, \vec{q}) \in G_2$, we put $G(s) := Z_G(s)$ and $G_2(a) := Z_{G_2}(a)$.

Notice that our Langlands parameters do not have additional data as in the usual Deligne-Langlands-Lusztig correspondence. This is because the (equivariant) fundamental groups of orbits are always trivial (c.f. Theorem 4.10). Instead, we have the following kind of difficulty:

Example C (Non-regular parameters). Let $G = Sp(4, \mathbb{C})$ and let $a = (\exp(r\epsilon_1 + (r + \pi\sqrt{-1})\epsilon_2), e^r, -e^r, -e^{2r}) \in T \times (\mathbb{C}^\times)^3$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Then, the number of $G_2(a)$-orbits in $\mathfrak{N}_2^a$ is eight, while the number of corresponding representations of $\mathbb{H}$ is six. (c.f. Enomoto [En06]) These orbits contain weight vectors of $\epsilon_1 + \epsilon_2$ or "ϵ_1 & ϵ_2".

Now we state the main theorem of this paper:

Theorem D (= Theorem 10.2)**.** *The set of G-conjugacy classes of admissible parameters is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}$-modules if q_2 is not a root of unity of order $\leq 2n$, and $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \leq m < n$.*

We treat a slightly more general case in Theorem 10.1 including Example C. Since the general condition is rather technical, we state only a part of it here.

By imposing an additional relation $q_0 + q_1 = 0$, the algebra $\mathbb{H}$ specializes to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two-parameters. (c.f. Remark 2.2.) Therefore, Theorem D also gives a definitive classification of simple $\mathbb{H}_B$-modules except for $-q_0^2 = q_2^m$ $(|m| < n)$ or q_2 is a root of unity of order $\leq 2n$.

Let us illustrate an example which (partly) explains the title "exotic":

Example E (Equal parameter case). Let $G = Sp(4, \mathbb{C})$. Let $s = \exp(r\epsilon_1 + r\epsilon_2) \in T$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Fix $a_0 = (s, e^{2r}) \in G \times \mathbb{C}^\times$ and $a = (s, e^r, -e^r, e^{2r}) \in G_2$. Let $\mathcal{N}$ be the nilpotent cone of G. Then, the sets of $G(s)$-orbits of $\mathcal{N}^{a_0}$ and $\mathfrak{N}_2^a$ are responsible for the usual and our exotic Deligne-Langlands correspondences. The number of $G(s)$-orbits in $\mathcal{N}^{a_0}$ is three. (Corresponding to root vectors of $\emptyset$, $2\epsilon_1$, and "$2\epsilon_1$ & $2\epsilon_2$") The number of $G(s)$-orbits in $\mathfrak{N}_2^a$ is four. (Corresponding to weight vectors of $\emptyset$, ϵ_1, $\epsilon_1 + \epsilon_2$, and "ϵ_1 & $\epsilon_1 + \epsilon_2$") On the other hand, the actual number of simple modules arising in this way is four (c.f. Ram [Ra01] and [En06]).

The organization of this paper is as follows:

In §1, we fix notation and introduce exotic nilcones and related varieties. In particular, we present geometric structures involved in our varieties as much as we need in the later sections. In §2, we prove Theorem A, which connects our varieties with an affine Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$. In order to simplify the study of representation theory of $\mathbb{H}$, we divide our varieties into a product of primitive ones in §3. In §4, we prove that the stabilizers of exotic nilpotent orbits are connected, which implies that "the Lusztig part" of the Deligne-Langlands-Lusztig parameter should be always trivial in our situation. Unfortunately, we have no nice parabolic subgroup as Kazhdan-Lusztig employed in [KL87]. We construct some explicit semisimple element out of each orbit in §5 for the sake of compensation. We introduce the notion of exotic Springer fibers and prove its odd-term vanishing result in §6, under the assumption that the parameters are sufficiently nice (including admissible case). Its proof essentially relies on the argument of §5. We define our standard modules as the total homology group of exotic Springer fibers in §7. At the same time, we present an induction

theorem, which claims that they behave well under inductions. In §8, we present an analogue of the Springer correspondence for exotic nilcones. In order to prove Theorem D, we still need two additional structural results. One is that our geometric structure is preserved by replacing the central character by a suitable real positive one. The other is that we can embed the corresponding finite Weyl group into the graded version of $\mathbb{H}$. Our proofs of both results essentially use admissibility of parameters. These results occupy §9. With the knowledge of all of the previous sections except for §7, we prove Theorem D in §10. The last section §11 concerns with applications, which are straight-forward consequences of Ginzburg theory assuming the results presented in earlier sections.

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1 Preparatory materials

Let $G := Sp(2n, \mathbb{C})$. Let B be a Borel subgroup of G. Let T be a maximal torus of B. Let $X^*(T)$ be the character group of T. Let R be the root system of (G, T) and let R^+ be its positive part defined by B. We embed R and R^+ into a n-dimensional Euclid space $\mathbb{E} = \oplus_i \mathbb{C}\epsilon_i$ with standard inner product as:

$$R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\} \subset \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\} = R \subset \mathbb{E}.$$

By the inner product, we identify ϵ_i with its dual basis. We put $\epsilon_i := -\epsilon_{-i}$ when $-n \le i < 0$. We put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$ or $2\epsilon_n$ $(i = n)$. Let W be the Weyl group of (G, T). For each α_i, we denote the reflection of $\mathbb{E}$ corresponding to α_i by s_i. Let $\ell : W \to \mathbb{Z}_{\ge 0}$ be the length function with respect to (B, T). We denote by $\dot{w} \in N_G(T)$ a lift of $w \in W$. For a subgroup $H \subset G$ containing T, we put ${}^wH := \dot{w}H\dot{w}^{-1}$. For a group H and its element s, we put $H(s) := Z_H(s)$. For a subset $S \subset H$, we put $H(S) := \cap_{s \in S} H(s)$. We denote the identity component of H by H°. We denote by $R(H)$ and $R(H)_s$ the representation ring of H and its localization along the evaluation at $s \in H$, respectively. For each $\alpha \in R$, we denote the corresponding one-parameter unipotent subgroup of G (with respect to T) by U_α. We define $\mathfrak{g}, \mathfrak{t}, \mathfrak{g}(s)$, etc... to be the Lie algebras of $G, T, G(s)$, etc..., respectively.

[1]**Note:** After the original version of this paper is circulated (in 2006, with different argument and weaker conclusion in Theorem D, and consequently give a classification of $\mathbb{H}$-modules only with a help of Lusztig's results [Lu88, Lu89, Lu95b]), there appeared two kinds of related works. One is the study of geometry which is connected to our nilcone by Achar-Henderson [AH08], Enomoto [En08], Finkelberg-Ginzburg-Travkin [FGT08], Springer [Sp07], Travkin [Tr08], and the other is the classification of tempered dual by Opdam and Solleveld [OS07, OS08, So07]. For the former, I have included explanations about the situation as much as I could in order to avoid potential problems. For the latter, we are preparing another paper [CK] in this direction.

For a T-module V, we define its weight λ-part (with respect to T) as $V[\lambda]$. We define the positive part V^+ and negative part V^- of V as

$$V^+ := \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0} R^+ - \{0\}} V[\lambda], \text{ and } V^- := \bigoplus_{\lambda \in \mathbb{Q}_{\leq 0} R^+ - \{0\}} V[\lambda],$$

respectively. We denote the set of T-weights of V by $\Psi(V)$.

In this paper, a segment is a set of integers I written as $I = [i_1, i_2] \cap \mathbb{Z}$ for some integers $i_1 \leq i_2$. By abuse of notation, we may denote I by $[i_1, i_2]$. For a segment I, we set $I^* := I$ (if $0 \notin I$) or $I - \{0\}$ (if $0 \in I$). We denote the absolute value function by $| \bullet | : \mathbb{C} \to \mathbb{R}_{\geq 0}$. We set $\Gamma_0 := 2\pi\sqrt{-1}\mathbb{Z} \subset \mathbb{C}$ and set $\exp : \mathbb{E} \to T$ to be the exponential map. We normalize the map exp so that $\ker \exp \cong \sum_{i=1}^n \Gamma_0 \epsilon_i$.

A variety in this paper is a quasi-projective reduced scheme of finite type over $\mathbb{C}$. Its points are closed points. If an algebraic group H acts on a variety $\mathcal{X}$, then we denote the stabilizer of the H-action at $x \in \mathcal{X}$ by $\mathsf{Stab}_H x$. For each $h \in H$, we denote by $\mathcal{X}^h$ the h-fixed point set of $\mathcal{X}$. For a variety $\mathcal{X}$, we denote by $H_\bullet(\mathcal{X})$ the Borel-Moore homology groups with coefficients $\mathbb{C}$.

1.1 Exotic nilpotent cones

Let $\ell = 0, 1$, or 2. We define $V_1 := \mathbb{C}^{2n}$ (vector representation) and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. These representations have B-highest weights ϵ_1 and $\epsilon_1 + \epsilon_2$, respectively. We put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation of $Sp(2n)$*. For $\ell \geq 1$, the set of non-zero weights of $\mathbb{V}_\ell$ is in one-to-one correspondence with R as

$$R \ni \begin{cases} \pm 2\epsilon_i \leftrightarrow \pm\epsilon_i & \in \Psi(V_1) \\ \pm\epsilon_i \pm \epsilon_j \leftrightarrow \pm\epsilon_i \pm \epsilon_j & \in \Psi(V_2) \end{cases}. \tag{1.1}$$

We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. We call this variety *the ℓ-exotic nilpotent cone*. By abuse of notation, we may denote the map $F_\ell \to \mathfrak{N}_\ell$ also by μ_ℓ.

Convention 1.1. For the sake of simplicity, we define objects F, $\mathfrak{N}$, $\mathbb{V}$, μ, etc... to be the objects F_ℓ, $\mathfrak{N}_\ell$, $\mathbb{V}_\ell$, μ_ℓ etc... with $\ell = 1$.

We summarize some basic geometric properties of $\mathfrak{N}_\ell$:

Theorem 1.2 (Geometric properties of $\mathfrak{N}_\ell$)**.** *We have the following:*

1. *The defining ideal of $\mathfrak{N}_\ell$ is generated by G-invariant polynomials of $\mathbb{C}[\mathbb{V}_\ell]$ without constant terms;*
2. *The variety $\mathfrak{N}_\ell$ is normal;*
3. *For $\ell = 1, 2$, the map μ_ℓ is a birational projective morphism onto $\mathfrak{N}_\ell$;*

4. Every fiber of the map μ_ℓ is connected;

In the below, we present properties which are valid only for the $\ell = 1$ case.

5. The set of G-orbits in $\mathfrak{N}_1$ is finite;

6. The map μ_1 is stratified semi-small with respect to the stratification of $\mathfrak{N}_1$ given by G-orbits.

Proof. The proof is given after Lemma 1.5 since we need extra notation. □

Lemma 1.3. *We have a natural identification*

$$F_\ell \cong \{(gB, X) \in G/B \times \mathbb{V}_\ell; X \in g\mathbb{V}_\ell^+\}.$$

Proof. Straightforward. □

Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We define a G_ℓ-action on $\mathfrak{N}_\ell$ as

$$G_\ell \times \mathfrak{N}_\ell \ni (g, q_{2-\ell}, \ldots, q_2) \times (X_{2-\ell} \oplus \cdots \oplus X_2) \mapsto (q_{2-\ell}^{-1} g X_{2-\ell} \oplus \cdots \oplus q_2^{-1} g X_2) \in \mathfrak{N}_\ell.$$

(Here we always regard $X_{2-\ell}, \ldots, X_1 \in V_1$ and $X_2 \in V_2$.) Similarly, we have a natural G_ℓ-action on F_ℓ which makes μ_ℓ a G_ℓ-equivariant map. We define $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. By Lemma 1.3, we have

$$Z_\ell := \{(g_1B, g_2B, X) \in (G/B)^2 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+\}.$$

We put

$$Z_\ell^{123} := \{(g_1B, g_2B, g_3B, X) \in (G/B)^3 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+ \cap g_3\mathbb{V}_\ell^+\}.$$

We define $p_i : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_iB, X) \in F_\ell$ and $p_{ij} : Z_\ell^{123} \ni (g_1B, g_2B, g_3B, X) \mapsto (g_iB, g_jB, X) \in Z_\ell$ ($i, j \in \{1, 2, 3\}$). We also put $\tilde{p}_i : F_\ell \times F_\ell \to F_\ell$ as the first and second projections ($i = 1, 2$). (Notice that the meaning of $p_i, \tilde{p}_i, p_{ij}$ depends on ℓ. The author hopes that there occurs no confusion on it.)

Lemma 1.4. *The maps p_i and p_{ij} $(1 \le i < j \le 3)$ are projective.*

Proof. The fibers of the above maps are given as the subsets of G/B defined by incidence relations. It is automatically closed, and we obtain the result. □

We have a projection

$$\pi_\ell : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_1B, g_2B) \in G/B \times G/B.$$

For each $w \in W$, we define a point $\mathsf{p}_w := B \times \dot{w}B \in G/B \times G/B$. This point is independent of the choice of $\dot{w}$. We put $\mathsf{O}_w := G\mathsf{p}_w \subset G/B \times G/B$. By the Bruhat decomposition, we have

$$G/B \times G/B = \bigsqcup_{w \in W} \mathsf{O}_w. \tag{1.2}$$

Lemma 1.5. *The variety Z_ℓ $(\ell = 1, 2)$ consists of $|W|$-irreducible components. Moreover, the dimensions of all of the irreducible components of Z are equal to $\dim F$.*

Proof. We first prove the assertion for $Z = Z_1$. By (1.2), the structure of Z is determined by the fibers over p_w. We have

$$\pi^{-1}(\mathsf{p}_w) = \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+.$$

By the dimension counting using (1.1), we deduce

$$\begin{aligned}\dim \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+ =& \dim V_1^+ \cap \dot{w}V_1^+ + \dim V_2^+ \cap \dot{w}V_2^+ \\ =& \#(R_l^+ \cap wR_l^+) + \#(R_s^+ \cap wR_s^+) = N - \ell(w),\end{aligned}$$

where $N := \dim \mathbb{V}^+ = \dim G/B$ and R_l^+, R_s^+ are the sets of long and short positive roots, respectively. As a consequence, we deduce

$$\dim \pi^{-1}(\mathsf{O}_w) = N + \ell(w) + N - \ell(w) = 2N.$$

Thus, each $\overline{\pi^{-1}(\mathsf{O}_w)}$ is an irreducible component of Z. Moreover, we have $\pi^{-1}(\mathsf{O}_1) \cong F$, which implies that the dimensions of irreducible components of Z are equal to $\dim F$.
Next, we prove the assertion for Z_2. By forgetting the first V_1-factor, we have a surjective map $\eta : Z_2 \to Z$. We have a surjective map $\eta' : Z \to Z_0$ given by forgetting the V_1-factor. The fiber of $(\eta' \circ \eta)$ at $x \in Z$ is isomorphic to the two-fold product of the fiber of η' at $\eta'(x)$. The latter fiber is isomorphic to the vector space $V_1^+ \cap gV_1^+$ when $\pi(x) = (1, g)\mathsf{p}_1$. Therefore, the preimage of each irreducible component of Z gives an irreducible component of Z_2. These irreducible components are distinct since their images under η must be distinct. Hence, the number of irreducible components of Z_2 is equal to the number of irreducible components of Z as desired. □

Proof of Theorem 1.2. The weight distribution of $\mathbb{V}^+$ and the Hesselink theory (c.f. [Po04] Theorem 1) claims that μ_ℓ gives a birational projective morphism onto an irreducible component of the Hilbert nilcone of $\mathbb{V}_\ell$. Here the Hilbert nilcone of $\mathbb{V}_\ell$ is an irreducible normal variety by Dadok-Kac [DK85] or Schwarz [Sc78]. In particular, our variety $\mathfrak{N}_\ell \subset \mathbb{V}_\ell$ is the Hilbert nilcone itself. Therefore, we obtain 1–3). 4) is an immediate consequence of 2), 3), and the Zariski main theorem (c.f. [CG97] 3.3.26). 5) is proved as a part of Proposition 1.16. We show 6). Let $\widehat{\mathbb{O}}$ be the inverse image of a G-orbit $G.X = \mathbb{O} \subset \mathfrak{N}$ under the map $\mu \circ p_2$. Then, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \widehat{\mathbb{O}}.$$

The dimension of the RHS is less than or equal to $\dim F$, which is the (constant) dimension of irreducible components of Z. In particular, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \mathfrak{N} = \dim F,$$

which implies that μ is semi-small. □

By a general result of [Gi97] p135 (c.f. [CG97] 2.7), the G_ℓ-equivariant K-group of Z_ℓ becomes an associative algebra via the map

$$\star : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(Z_\ell) \ni ([\mathcal{E}], [\mathcal{F}]) \mapsto \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (p_{13})_* (p_{12}^* \mathcal{E} \otimes^{\mathbb{L}} p_{23}^* \mathcal{F})] \in K^{G_\ell}(Z_\ell).$$

Moreover, the G_ℓ-equivariant K-group of F_ℓ becomes a representation of $K^{G_\ell}(Z_\ell)$ as

$$\circ : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(F_\ell) \ni ([\mathcal{E}], [\mathcal{K}]) \mapsto \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{E} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{K})] \in K^{G_\ell}(F_\ell).$$

Here we regard $\mathcal{E}$ as a sheaf over $F_\ell \times F_\ell$ via the natural embedding $Z_\ell \subset F_\ell \times F_\ell$.

1.2 Definition of parameters

In this subsection, we present the definitions of parameters which we need in the sequel. First, we put $a_0 := (1, 1, -1, 1) \in G_2$. (The value a_0 is special in the sense it naturally gives the Weyl group of type C in our framework. C.f. §8)

Definition 1.6 (Configuration of semisimple elements)**.**

1) An element $a = (s, q_0, q_1, q_2) \in G_2$ is called pre-admissible iff s is semisimple, $q_0 \neq q_1$, q_2 is not a root of unity of order $\leq 2n$.

2) An element $a \in G_2$ is called finite if $\mathfrak{N}_2^a$ has only finitely many $G_2(a)$-orbit.

3) A pre-admissible element $a = (s, q_0, q_1, q_2)$ is called admissible if $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \leq m < n$.

For a pre-admissible element $a = (s, q_0, q_1, q_2)$, we put

$$\mathbb{V}_2^a = V_1^{(s,q_0)} \oplus V_1^{(s,q_1)} \oplus V_2^{(s,q_2)} \subset V_1 \oplus V_1 \oplus V_2 = \mathbb{V}_2.$$

In the below, we may denote $(q_0, q_1, q_2) \in (\mathbb{C}^\times)^3$ by $\vec{q}$ for the sake of simplicity.

Let $a = (s, \vec{q}) \in G_2$ be a pre-admissible element such that $s \in T$. We sometimes denote it as

$$s = \exp\left(\sum_{i=1}^{n} \log_i(s) \epsilon_i\right) \in \exp(\mathbb{E}) \cong T,$$

where $\log_i(s) \in \mathbb{C}$.

Remark 1.7. The values of $\log_i(s)$ are determined modulo Γ_0. Here we understand that $\log_i(s)$ is a fixed choice of a representative in $\log_i(s) + \Gamma_0$.

Definition 1.8 (Admissible parameters)**.**

1) A pre-admissible parameter is a pair

$$\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2) \in G_2 \times \mathfrak{N}_1$$

such that a is pre-admissible, $(s - q_0)(s - q_1) X_1 = 0$, and $s X_2 = q_2 X_2$;

For a pre-admissible $a \in G_2$, we denote by Λ_a the set of $G(s)$-conjugacy classes of pre-admissible parameters of the form (a, Y), where $Y \in \mathbb{V}$.

2) A pre-admissible parameter $\nu = (a, X)$ is called admissible if a is admissible.

1.3 Orbit structures arising from $\mathfrak{N}_\ell$

In the below, we fix vectors in V_1 and V_2 as follows:

- For each $i \in [-n, n]^*$, we define $0 \neq x_i \in V_1$ as a non-zero vector of weight ϵ_i;
- For each distinct $i, j \in [-n, n]^*$, we define $y_{ij} \in V_2$ to be a non-zero vector of weight $\epsilon_i - \epsilon_j$.

The following is a slight enhancement of the good basis of Ohta [Oh86] (1.3).

Definition 1.9 (Signed partitions)**.** Let $\mathbf{J} := \{J_1, J_2, \ldots\}$ be a collection of sequence of elements of $[-n, n]^*$. (I.e. each $J_k \in \mathbf{J}$ is a sequence $(J_k^1, J_k^2, \ldots)$ in $[-n, n]^*$.) We put $J_k^+ = (|J_k^1|, |J_k^2|, \ldots)$ for each $k = 1, 2, \ldots$. We call $\mathbf{J}$ a signed partition of n if and only if $\{J_1^+, J_2^+, \ldots\}$ gives a subdivision of $[1, n]$. I.e. we have

$$[1, n] = \bigcup_{k \geq 1} J_k^+ = \bigcup_{k \geq 1} \{|j|; j \in J_k\} \text{ and } J_k^+ \cap J_{k'}^+ = \emptyset \text{ for } k \neq k'.$$

For each member J of a signed partition $\mathbf{J}$, we define a subtorus

$$T_J := \exp \sum_{i \in J} \mathbb{C}\epsilon_i \subset T.$$

Let $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of n. Then, we regard it as a signed partition by setting

$$J_j^i := i + \sum_{k=1}^{j-1} \lambda_k \text{ if } \lambda_j \neq 0 \text{ and } 1 \leq i \leq \lambda_j.$$

Definition 1.10 (Foot functions)**.** Let $\ell = 0, 1,$ or 2. A collection of ℓ-tuple of functions $\delta_k : [-n, n]^* \to \{0, 1\}$ for $1 \leq k \leq \ell$ is called a ℓ-foot function of n. We denote a ℓ-foot function $\{\delta_k\}_{k=1}^{\ell}$ by $\vec{\delta}$.

Notice that Definition 1.10 claims that $\vec{\delta} = \emptyset$ when $\ell = 0$.

Definition 1.11 (Marked partitions, blocks, and normal forms)**.** Let ℓ be as in Definition 1.10. We refer a pair $\sigma = (\mathbf{J}, \vec{\delta})$ consisting of a signed partition and a ℓ-foot function of n as a ℓ-marked partition if the following condition holds:

- For each $J \in \mathbf{J}$ and $m = 1, \ldots, \ell$, we have

$$\#\{j \in J; \delta_m(j) = 1\} + \#\{j \in J; \delta_m(-j) = 1\} \leq 1.$$

For each $J \in \mathbf{J}$, we define the ℓ-block $\mathbf{v}_\sigma^J = \mathbf{v}_{\sigma,1}^J + \mathbf{v}_{\sigma,2}^J \in \mathbb{V}$ associated to $(\sigma, J) = (\sigma, \{J^1, J^2, \ldots\})$ as:

$$\mathbf{v}_{\sigma,1}^J := \sum_{j \in J} \sum_{k=1}^{\ell} (\delta_k(j) x_j + \delta_k(-j) x_{-j}) \in V_1$$

$$\mathbf{v}_{\sigma,2}^J := \sum_{j \geq 1} y_{J^j, J^{j+1}} \in V_2,$$

where we regard $y_{J_k^j, J_k^{j+1}} \equiv 0$ whenever J_k nor J_k^{j+1} is non-existent.
A ℓ-normal form $\mathbf{v}_\sigma = \mathbf{v}_{\sigma,1} + \mathbf{v}_{\sigma,2} \in \mathbb{V}$ associated to σ is defined as:

$$\mathbf{v}_{\sigma,1} := \sum_{J \in \mathbf{J}} \mathbf{v}_{\sigma,1}^J \in V_1, \text{ and } \mathbf{v}_{\sigma,2} := \sum_{J \in \mathbf{J}} \mathbf{v}_{\sigma,2}^J \in V_2.$$

Remark 1.12. We regard that ℓ-normal forms are elements of $\mathbb{V} = \mathbb{V}_1$, regardless the value of ℓ.

Definition 1.13 (Strict normal forms)**.** A ℓ-marked partition $\sigma = (\mathbf{J}, \vec{\delta})$ is called strict if and only if the following four conditions hold:

1. $\mathbf{J}$ is obtained from a partition λ of n;

2. We have $\delta_2 \equiv 0$ and $\delta_1(j) = 0$ for every $j \in [-n, -1]$;

Before stating the rest of the conditions, we introduce extra notation. Assume the above two conditions. If we have $\delta_1(j) = 1$ for $j \in J$, then we set $\underline{\#} J := \#\{j' \in J; j' \leq j\}$ and $\overline{\#} J := \#\{j' \in J; j' > j\}$.

3. Let $k < m$ be two integers and let $\mathbf{J} = \{J_1, J_2, \ldots\}$. Then, we have $\delta_1|_{J_m} \equiv 0$ if $\# J_k = \# J_m$;

4. Let $J, J' \in \mathbf{J}$ be a pair such that $\delta_1(j) = 1 = \delta_1(j')$ for some $j \in J$ and $j' \in J'$. If $\# J > \# J'$, then we have
$$\underline{\#} J > \underline{\#} J' \text{ and } \overline{\#} J > \overline{\#} J'.$$

Conditions are not applicable when δ_2 or δ_1 are non-existent. Notice that only the first condition survives when $\ell = 0$. A normal form attached to a strict ℓ-marked partition is called a ℓ-strict normal form.

In the below, we refer foot functions, blocks, normal forms..., to be the 1-foot functions, 1-blocks, 1-normal forms..., respectively. Moreover, we naturally identify strict 1-normal forms and strict 2-normal forms since $\delta_2 \equiv 0$ for 2-strict marked partitions.

Let $\mathsf{Irrep} W$ be the set of isomorphism classes of irreducible W-modules.

Theorem 1.14 (Orbit description of $\mathfrak{N}_1$)**.** *We have:*

1. *The set of strict* 1*-normal forms is in one-to-one correspondence with the the set of G-orbits of $\mathfrak{N}_1$;*

2. *We have $\#(G \backslash \mathfrak{N}_1) = \# \mathsf{Irrep} W$;*

3. *For each $X \in \mathfrak{N}_1$, the group $\mathsf{Stab}_G X$ is connected.*

Remark 1.15. The original form of the proof of Theorem 1.14 (in [Ka06b]) employs explicit calculation using basis. In the meantime, Springer [Sp07] gives a base-free proof (with stronger consequences). The proof given here is somewhat the mixture of the both, which the author gives it for the sake of completeness. Note that the closure relation of the orbits of $\mathfrak{N}_1$ is calculated by Achar-Henderson [AH08].

The proof of Theorem 1.14 is obtained as a combination of Proposition 4.5 and Theorem 8.3 by using the knowledge of the following:

Proposition 1.16 (Weak version of Theorem 1.14)**.** *We have:*

1. *Each G-orbit of $\mathfrak{N}_1$ contains a strict normal form;*
2. *The number of elements of the set of strict marked partitions is less than or equal to $\#\mathsf{Irrep}W$.*

Proof. By a result of Ohta-Sekiguchi [Se84, Oh86], the set of strict 0-marked partitions are in one-to-one correspondence with the set of G-orbits of $\mathfrak{N}_0$ via the assignment $\sigma \mapsto G\mathbf{v}_\sigma$. We have

$$\mathbb{C}[\mathbb{V}]^G \cap \mathbb{C}[\mathbb{V}_0] = \mathbb{C}[\mathbb{V}_0]^G,$$

which gives the natural projection map

$$\mathfrak{N}_1 \longrightarrow \mathfrak{N}_0$$

obtained from the natural projection $\mathbb{V}_1 \to \mathbb{V}_0$. (In fact we have $\mathbb{C}[\mathbb{V}]^G = \mathbb{C}[\mathbb{V}_0]^G$. But this fact is not used here.) It follows that each orbit of $\mathfrak{N}_1$ contains a vector of type $v = v_1 \oplus \mathbf{v}_{2,\lambda}$, where λ is a partition of n regarded as a strict 0-marked partition $(\mathbf{J}, \emptyset)$ in a natural way.
Consider the action of

$$G' := Sp(2\lambda_1) \times Sp(2\lambda_2) \times \cdots \subset Sp(2n),$$

which are embedded so that $T \subset G'$ and V_1 restricted to G' has the form

$$\mathsf{Res}^G_{G'} V_1 = \bigoplus_{k \ge 1} V_1^k$$

such that V_1^k is a vector representation of $Sp(2\lambda_k)$ with T-weights $\pm\epsilon_i$ for $i = 1 + \sum_{j=1}^{k-1} \lambda_j, \ldots, \lambda_k + \sum_{j=1}^{k-1} \lambda_j$.
Let ω be the symplectic form on V_1 which is preserved by G. For each k, we put $\omega_k := \omega|_{V_1^k}$. We have $v = \sum_{k \ge 1} v_k$, where $v_k = v_{1,k} \oplus \mathbf{v}_{2,\lambda}^{J_k} \in V_1^k \oplus \wedge^2 V_1^k$. We consider an identification of y_{ij} $(i, j \in J_k)$ with a matrix such that $y_{ij} x_k = c_{ij} x_i$ $(k = j)$, $-c_{ij} x_j$ $(k = -i)$, 0 (otherwise), for some $c_{ij} \in \mathbb{C}$. We arrange $\{c_{ij}\}_{i,j}$ so that $\wedge^2 V_1^k$ is $Sp(2\lambda_k)$-equivariantly identified with the subset of $\mathrm{End} V_1^k$ such that

$$\omega_k(y_{ij}x, x') = \omega_k(x, y_{ij}x') \text{ for each } x, x' \in V_1^k \text{ and } i, j \in (J_k \cup -J_k).$$

The Ohta-Sekiguchi result asserts that this gives an identification of $Sp(2\lambda_k)\mathbf{v}_{2,\lambda}^{J_k}$ and the set of linear nilpotent endomorphisms on V_1^k of maximal rank ($= \dim V_1^k - 2$) which preserve ω_k. Since $v_{1,k}$ can be complemented to a suitable choice of a standard basis of V_1^k (as a symplectic vector space), we deduce that a suitable change of symplectic basis makes $v_{1,k}$ into one of x_i $(i > 0)$. This implies that v can be transformed into a 1-normal form associated to $(\mathbf{J}, \delta_1) = (\lambda, \delta_1)$ which satisfies 1.13 1) and 2).
Now for each $k < k'$, we examine the $Sp(2\lambda_k + 2\lambda_{k'})$-orbit which contains $v_k + v_{k'} \in V_1^k \oplus V_1^{k'}$. We have $\lambda_k \ge \lambda_{k'}$ by 1.13 1). We put $\xi := \mathbf{v}_{2,\sigma}^{J_k} + \mathbf{v}_{2,\sigma}^{J_{k'}}$. The $Sp(2\lambda_k + 2\lambda_{k'})$-conjugacy class of ξ is the set of nilpotent endomorphisms of $V_1^k \oplus V_1^{k'}$ which preserve ω and have $(\lambda_k, \lambda_k, \lambda_{k'}, \lambda_{k'})$ as its (nilpotent) Jordan

form. If $v_{1,k} = 0$ or $v_{1,k'} = 0$ hold, then 1.13 3) and 4) are satisfied for the pair $(J_k, J_{k'})$. Hence, we assume $v_{1,k} \neq 0 \neq v_{1,k'}$ in the below. We have

$$\xi^{\underline{\#} J_k} v_{1,k} = 0, \xi^{\underline{\#} J_k - 1} v_{1,k} \neq 0, \text{ and } \xi^{\underline{\#} J_{k'}} v_{1,k'} = 0, \xi^{\underline{\#} J_{k'} - 1} v_{1,k'} \neq 0$$
$$v_{1,k} \in \mathrm{Im} \xi^{\overline{\#} J_k}, v_{1,k} \notin \mathrm{Im} \xi^{\overline{\#} J_k + 1}, \text{ and } v_{1,k'} \in \mathrm{Im} \xi^{\overline{\#} J_{k'}}, v_{1,k'} \notin \mathrm{Im} \xi^{\overline{\#} J_{k'} + 1}.$$

If $\underline{\#} J_k \leq \underline{\#} J_{k'}$ or $\overline{\#} J_k \leq \overline{\#} J_{k'}$ holds, then we can regard $v_{1,k} + v_{1,k'}$ as a part of a standard basis of a $(\mathbf{v}_{2,\sigma})$-stable symplectic subspace of $(V_1^{k'} \oplus V_1^k)$ isomorphic to $V_1^{k'}$ or V_1^k, respectively. When $\lambda_k = \lambda_{k'}$, we use this to change our normal form so that the corresponding marked partition satisfies 1.13 3). When $\lambda_k > \lambda_{k'}$, we use this to transform our marked partition into another marked partition which satisfies 1.13 4) for the pair $(J_k, J_{k'})$. If we make changes to our marked partition in one of the above two procedures, then $\mathbf{J}$ is unchanged, δ_1 on one of $\{J_k, J_{k'}\}$ is unchanged, but δ_1 on the other becomes 0. By repeating these procedures for every possible pair $k < k'$, we complete the proof of the first assertion.
For the second assertion, recall that $\mathsf{Irrep} W$ is parametrized by the set of ordered pair of partitions (λ^1, λ^2) which sum up to n. We define two-partitions out of a strict marked partition σ as

$$\lambda^1_k + \lambda^2_k = \lambda_k, \text{ and } \lambda^2_k := \begin{cases} \underline{\#} J_k & (\mathbf{v}_{1,\sigma}^{J_k} \neq 0) \\ \max\{0, \underline{\#} J_{k'}, \lambda_k - \overline{\#} J_{k''}; k' > k > k''\} & (otherwise) \end{cases}$$

for each k, where the set we choose its maximal is formed only from these $J_{k'}$ and $J_{k''}$ for which $\overline{\#}$ and $\underline{\#}$ are defined. It is clear that two sequences λ^1, λ^2 sum up to n. By 1.13 4), we deduce that

$$\lambda_k - \overline{\#} J_{k''} < \underline{\#} J_k \text{ (this is equivalent to } \overline{\#} J_{k''} > \overline{\#} J_k)$$

holds for $k'' < k$ (such that $\overline{\#}$ and $\underline{\#}$ are defined for both J_k and $J_{k''}$). It follows that λ^2 is a partition. (I.e. $\{\lambda^2_i\}_i$ is a decreasing sequence.) By the symmetry of $\overline{\#}$ and $\underline{\#}$ in 1.13, we conclude that λ^1 is also a partition.
Therefore, it suffices to prove that the pairs of partitions formed by strict marked partitions are equal only if the marked partitions are equal. (Since this gives the injectivity of the above assignment.) For this, we assume that two strict marked partitions $\sigma = (\mathbf{J}, \delta_1)$ and $\sigma' = (\mathbf{J}', \delta_1')$ gives the same pair (λ^1, λ^2) to deduce contradiction. We can assume that $\mathbf{J} = \mathbf{J}'$ since $\lambda = \lambda^1 + \lambda^2$. Hence, their difference is concentrated in their foot function. By 1.13 3) and 4), we deduce that the foot functions are non-trivial on $J_k = J'_k$ if and only if

$$\lambda^2_k \neq \max\{\lambda^2_j, \lambda_k - \lambda^1_i; \lambda_j \neq \lambda_k \neq \lambda_i, i < k < j\}$$

and $\lambda_k \neq \lambda_{k-1}$. Moreover, the value of the foot functions on J_k are determined by the value of λ^2_k if they are non-trivial. Since this system has a unique solution, we deduce $\sigma = \sigma'$, which is contradiction. Thus, the pair of partitions recovers a strict marked partition uniquely, which completes the proof of the second assertion. $\square$

Theorem 1.17 (Orbit structure of $\mathfrak{N}^a_2$)**.** *Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter. Then, there exists $g \in G$ such that:*

$$gsg^{-1} \in T \textit{ and } gX \textit{ is a normal form.}$$

Proof. Postponed to §4. □

We have a natural W-action $\cdot$ on $[-n,n]^*$ by setting

$$s_i \cdot j := \begin{cases} \pm(i+1) & (j=\pm i) \\ \pm i & (j=\pm(i+1)) \\ j & (otherwise) \end{cases} \text{ for } i=1,\ldots,n-1, \text{ and } s_n \cdot j = \begin{cases} -j & (j=\pm n) \\ j & (otherwise) \end{cases}.$$

Using this, we define the W-action $\cdot$ on the set of ℓ-marked partitions as:
For $w \in W$ and $\sigma = (\mathbf{J}, \vec{\delta}) = (\{J_1, J_2, \ldots,\}, \{\delta_1, \ldots, \delta_\ell\})$, we set

$$w \cdot \sigma := (\{w \cdot J_1, w \cdot J_2, \ldots,\}, \{w \cdot \delta_1, \ldots, w \cdot \delta_\ell\}),$$

where we set

$$w \cdot (J_1^1, J_1^2, \ldots) = ((w \cdot J)_1^1, (w \cdot J)_1^2, \ldots) := (w \cdot J_1^1, w \cdot J_1^1, \ldots)$$

and $w \cdot \delta_k(j) := \delta_k(w^{-1} \cdot j)$. Notice that we have $w \cdot J_k = (w \cdot J)_k$ and $w \cdot J_k^j = (w \cdot J)_k^j$ for every k, j in this action.

Lemma 1.18. *Let $\sigma = (\mathbf{J}, \vec{\delta})$ be a marked partition which is a W-translation of a strict marked partition. Then, we have*

$$\mathbb{C}^\times \mathbf{v}_{1,\sigma} \oplus \mathbb{C}^\times \mathbf{v}_{2,\sigma} \subset T\mathbf{v}_\sigma, \text{ and } \mathbb{C}^\times \mathbf{v}_{1,\sigma}^J \oplus \mathbb{C}^\times \mathbf{v}_{2,\sigma}^J \subset T_J \mathbf{v}_\sigma^J \text{ for each } J \in \mathbf{J}.$$

Proof. Since $T_J \cap T_{J'} = \{1\}$, it suffices to prove the second assertion. Let $\mathbf{v}_\sigma^J = \sum_{\xi \in \Xi} v_\xi$ be the T-eigen-decomposition of $\mathbf{v}_\sigma^J$. Then, we have $\# \Xi = \# J$ and $\dim T_J = \# J$. Moreover, the weights appearing in Ξ are linearly independent. Hence, we have the scalar multiplications of each v_ξ, which implies the result. □

Corollary 1.19 (of the proof of Lemma 1.18)**.** *Let σ be a strict marked partition. Let $w \in W$. Then, we have $\mathbf{v}_{w \cdot \sigma} \in G\mathbf{v}_\sigma$.* □

1.4 Structure of simple modules

We put $T_\ell := T \times (\mathbb{C}^\times)^{\ell+1}$. Let $a \in T_\ell$. Let $\mu^a : F_\ell^a \to \mathfrak{N}_\ell^a$ denote the restriction of μ_ℓ to a-fixed points.
We review the convolution realization of simple modules in our situation. The detailed constructions are found in [CG97] 5.11, 8.4 or [Gi97] §5. For its variant, see [Jo98].
The properties we used to apply the Ginzburg theory are: 1) $Z_\ell = F_\ell \times_{\mathfrak{N}_\ell} F_\ell$; 2) F_ℓ is smooth; 3) μ_ℓ is projective; 4) $R(G_\ell) \subset K^{G_\ell}(Z_\ell)$ is central; and 5) $H_\bullet(Z_\ell)$ is spanned by algebraic cycles.
Let $\mathbb{C}_a$ be the unique residual field of $\mathbb{C} \otimes_\mathbb{Z} R(G_\ell)_a$ or $\mathbb{C} \otimes_\mathbb{Z} R(T_\ell)_a$. The Thomason localization theorem yields ring isomorphisms

$$\mathbb{C}_a \otimes_{R(G_\ell)} K^{G_\ell}(Z_\ell) \xrightarrow{\cong} \mathbb{C}_a \otimes_{R(G_\ell(a))} K^{G_\ell(a)}(Z_\ell^a) \xrightarrow{\cong} \mathbb{C}_a \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^a).$$

Moreover, we have the Riemann-Roch isomorphism

$$\mathbb{C}_a \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^a) \cong K(Z_\ell^a) \xrightarrow{RR} H_\bullet(Z_\ell^a) \cong \mathrm{Ext}^\bullet(\mu_*^a \mathbb{C}_{F_\ell^a}, \mu_*^a \mathbb{C}_{F_\ell^a}).$$

By the equivariant Beilinson-Bernstein-Deligne (-Gabber) decomposition theorem (c.f. Saito [Sa88] 5.4.8.2), we have

$$\mu^a_* \mathbb{C}_{F^a_\ell} \cong \bigoplus_{\mathbb{O} \subset \mathfrak{N}^a_\ell, \chi, d} L_{\mathbb{O},\chi,d} \boxtimes IC(\mathbb{O}, \chi)[d],$$

where $\mathbb{O} \subset \mathfrak{N}^a_\ell$ is a $G(s)$-stable subset such that μ^a is locally trivial along $\mathbb{O}$, χ is an irreducible local system on $\mathbb{O}$, d is an integer, $L_{\mathbb{O},\chi,d}$ is a finite dimensional vector space, and $IC(\mathbb{O}, \chi)$ is the minimal extension of χ. Moreover, the set of $\mathbb{O}$'s such that $L_{\mathbb{O},\chi,d} \neq 0$ (for some χ and d) forms a subset of an algebraic stratification in the sense of [CG97] 3.2.23. It follows that:

Theorem 1.20 (Ginzburg [Gi97] Theorem 5.2)**.** *The set of simple modules of $K^{G_\ell}(Z_\ell)$ for which $R(G_\ell)$ acts as the evaluation at a is in one-to-one correspondence with the set of isomorphism classes of irreducible $G_\ell(a)$-equivariant perverse sheaves appearing in $\mu^a_* \mathbb{C}_{F^a_\ell}$ (up to degree shift).* □

2 Hecke algebras and exotic nilpotent cones

We retain the setting of the previous section. We put $\boldsymbol{G} = G_2$, $\boldsymbol{T} := T_2$, $\boldsymbol{G} := F_2$, $\boldsymbol{\mu} := \mu_2$, $\boldsymbol{Z} := Z_2$, and $\boldsymbol{\pi} := \pi_2$. Most of the arguments in this section are exactly the same as [CG97] 7.6 if we replace $\boldsymbol{G}$ by $G \times \mathbb{C}^\times$, $\mathfrak{N}_2$ by the usual nilpotent cone, $\boldsymbol{\mu}$ by the moment map, $\boldsymbol{F}$ by the cotangent bundle of the flag variety, and $\boldsymbol{Z}$ by the Steinberg variety. Therefore, we frequently omit the detail and make pointers to [CG97] 7.6 in which the reader can obtain a correct proof merely replacing the meaning of symbols as mentioned above.

We put $\mathcal{A}_\mathbb{Z} := \mathbb{Z}[\mathbf{q}_0^{\pm1}, \mathbf{q}_1^{\pm1}, \mathbf{q}_2^{\pm1}]$ and $\mathcal{A} := \mathbb{C} \otimes_\mathbb{Z} \mathcal{A}_\mathbb{Z} = \mathbb{C}[\mathbf{q}_0^{\pm1}, \mathbf{q}_1^{\pm1}, \mathbf{q}_2^{\pm1}]$.

Definition 2.1 (Hecke algebras of type $C_n^{(1)}$)**.** A Hecke algebra of type $C_n^{(1)}$ with three parameters is an associative algebra $\mathbb{H}$ over $\mathcal{A}$ generated by $\{T_i\}_{i=1}^n$ and $\{e^\lambda\}_{\lambda \in X^*(T)}$ subject to the following relations:

(Toric relations) For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ (and $e^0 = 1$);

(The Hecke relations) We have

$$(T_i + 1)(T_i - \mathbf{q}_2) = 0 \ (1 \le i < n) \text{ and } (T_n + 1)(T_n + \mathbf{q}_0\mathbf{q}_1) = 0;$$

(The braid relations) We have

$$\begin{gathered} T_iT_j = T_jT_i \text{ (if } |i-j| > 1), \ (T_nT_{n-1})^2 = (T_{n-1}T_n)^2, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \text{ (if } 1 \le i < n-1); \end{gathered}$$

(The Bernstein-Lusztig relations) For each $\lambda \in X^*(T)$, we have

$$T_i e^\lambda - e^{s_i\lambda} T_i = \begin{cases} (1-\mathbf{q}_2)\frac{e^\lambda - e^{s_i\lambda}}{e^{\alpha_i}-1} & (i \neq n) \\ \frac{(1+\mathbf{q}_0\mathbf{q}_1)-(\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n}}{e^{\alpha_n}-1}(e^\lambda - e^{s_n\lambda}) & (i = n) \end{cases}.$$

Remark 2.2. **1)** The standard choice of parameters (t_0, t_1, t_n) is: $t_1^2 = \mathbf{q}_2$, $t_n^2 = -\mathbf{q}_0\mathbf{q}_1$, and $t_n(t_0 - t_0^{-1}) = (\mathbf{q}_0 + \mathbf{q}_1)$. This yields

$$T_n e^\lambda - e^{s_n\lambda} T_n = \frac{1 - t_n^2 - t_n(t_0 - t_0^{-1})e^{\epsilon_n}}{e^{2\epsilon_n} - 1}(e^\lambda - e^{s_n\lambda});$$

2) If $n = 1$, then we have $T_1 = T_n$ in Definition 2.1. In this case, we have $\mathbb{H} \cong \mathbb{C}[\mathbf{q}_2^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{H}_0$, where $\mathbb{H}_0$ is the Hecke algebra of type $A_1^{(1)}$ with two-parameters $(\mathbf{q}_0, \mathbf{q}_1)$;
3) An extended Hecke algebra of type $B_n^{(1)}$ with two-parameters considered in [En06] is obtained by requiring $\mathbf{q}_0 + \mathbf{q}_1 = 0$. An equal parameter extended Hecke algebra of type $B_n^{(1)}$ is obtained by requiring $\mathbf{q}_0 + \mathbf{q}_1 = 0$ and $\mathbf{q}_1^2 = \mathbf{q}_2$. An equal parameter Hecke algebra of type $C_n^{(1)}$ is obtained by requiring $\mathbf{q}_2 = -\mathbf{q}_0\mathbf{q}_1$ and $(1 + \mathbf{q}_0)(1 + \mathbf{q}_1) = 0$.

For each $w \in W$, we define two closed subvarieties of $\boldsymbol{Z}$ as

$$Z_{\leq w} := \boldsymbol{\pi}^{-1}(\overline{\mathsf{O}_w}) \text{ and } Z_{<w} := Z_{\leq w} \backslash \boldsymbol{\pi}^{-1}(\mathsf{O}_w).$$

Let $\lambda \in X^*(T)$. Let $\mathcal{L}_\lambda$ be the pullback of the line bundle $G \times^B \lambda^{-1}$ over G/B to $\boldsymbol{F}$. Clearly $\mathcal{L}_\lambda$ admits a $\mathbf{G}$-action by letting $(\mathbb{C}^\times)^3$ act on $\mathcal{L}_\lambda$ trivially. We denote the operator $[\tilde{p}_1^* \mathcal{L}_\lambda \otimes^{\mathbb{L}} \bullet]$ by e^λ. By abuse of notation, we may denote $\mathsf{e}^\lambda(1)$ by e^λ (in $K^{\boldsymbol{G}}(\boldsymbol{Z})$). Let $\mathsf{q}_0 \in R(\{1\} \times \mathbb{C}^\times \times \{1\} \times \{1\}) \subset R(\boldsymbol{G})$, $\mathsf{q}_1 \in R(\{1\} \times \{1\} \times \mathbb{C}^\times \times \{1\}) \subset R(\boldsymbol{G})$, and $\mathsf{q}_2 \in R(\{1\} \times \{1\} \times \{1\} \times \mathbb{C}^\times) \subset R(\boldsymbol{G})$ be the inverse of degree-one characters. (I.e. q_2 corresponds to the inverse of the scalar multiplication on V_2.) By the operation e^λ and the multiplication by q_i, each of $K^{\boldsymbol{G}}(Z_{\leq w})$ admits a structure of $R(\boldsymbol{T})$-modules.

Each $Z_{\leq w} \backslash Z_{<w}$ is a $\boldsymbol{G}$-equivariant vector bundle over an affine fibration over G/B via the composition of $\boldsymbol{\pi}$ and the second projection. Therefore, the cellular fibration Lemma (or the successive application of localization sequence) yields:

Theorem 2.3 (c.f. [CG97] 7.6.11)**.** *We have*

$$K^{\boldsymbol{G}}(Z_{\leq w}) = \bigoplus_{v \in W; \mathsf{O}_v \subset \overline{\mathsf{O}}_w} R(\boldsymbol{T})[\mathcal{O}_{Z_{\leq v}}].$$

For each $i = 1, 2, \ldots, n$, we put $\mathbb{O}_i := \overline{\boldsymbol{\pi}^{-1}(\mathsf{O}_{s_i})}$. We define $\tilde{T}_i := [\mathcal{O}_{\mathbb{O}_i}]$ for each $i = 1, \ldots, n$.

Theorem 2.4 (c.f. Proof of [CG97] 7.6.12)**.** *The set* $\{[\mathcal{O}_{Z_{\leq 1}}], \tilde{T}_i, \mathsf{e}^\lambda; 1 \leq i \leq n, \lambda \in X^*(T)\}$ *is a generator set of* $K^{\boldsymbol{G}}(\boldsymbol{Z})$ *as* $\mathcal{A}_{\mathbb{Z}}$*-algebras.*

Proof. The tensor product of structure sheaves corresponding to vector subspaces of a vector space is the structure sheaf of their intersection. Taking account into that, the proof of the assertion is exactly the same as [CG97] 7.6.12. □

By the Thom isomorphism, we have an identification

$$K^{\boldsymbol{G}}(\boldsymbol{F}) \cong K^{\boldsymbol{G}}(G/B) \cong R(\boldsymbol{T}) = \mathcal{A}_{\mathbb{Z}}[T]. \tag{2.1}$$

We normalize the images of $[\mathcal{L}_\lambda]$ and q_i $(i = 0, 1, 2)$ under (2.1) as e^λ and $\mathbf{q}_i$, respectively.

Theorem 2.5 (c.f. [CG97] Claim 7.6.7)**.** *The homomorphism*

$$\circ : K^{\boldsymbol{G}}(\boldsymbol{Z}) \longrightarrow \mathrm{End}_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(\boldsymbol{F})$$

is injective. □

Proposition 2.6. *We have*

1. $[\mathcal{O}_{Z_{\le 1}}] = 1 \in \mathrm{End}_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(\boldsymbol{F})$;

2. $\tilde{T}_i \circ e^{\lambda} = (1 - \mathbf{q}_2 e^{\alpha_i}) \frac{e^{\lambda} - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}$ *for every* $\lambda \in X^*(T)$ *and every* $1 \le i < n$;

3. $\tilde{T}_n \circ e^{\lambda} = (1 - \mathbf{q}_0 e^{\frac{1}{2}\alpha_n})(1 - \mathbf{q}_1 e^{\frac{1}{2}\alpha_n}) \frac{e^{\lambda} - e^{s_n \lambda - \alpha_n}}{1 - e^{-\alpha_n}}$ *for every* $\lambda \in X^*(T)$.

Proof. The component $Z_{\le 1}$ is equal to the diagonal embedding of $\boldsymbol{F}$. In particular, both of the first and the second projections give isomorphisms between $Z_{\le 1}$ and $\boldsymbol{F}$. It follows that

$$\begin{aligned}[\mathcal{O}_{Z_{\le 1}}] \circ [\mathcal{L}_\lambda] &= \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (p_1)_* \left(\mathcal{O}_{Z_{\le 1}} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{L}_\lambda\right)] \\ &= [\mathbb{R}^0 (p_1)_* \left(\mathcal{O}_{Z_{\le 1}} \otimes \tilde{p}_2^* \mathcal{L}_\lambda\right)] = [\mathcal{L}_\lambda],\end{aligned}$$

which proves 1). For each $i = 1, \ldots, n$, we define $\mathbb{V}^+(i) := \mathbb{V}_2^+ \cap \dot{s}_i \mathbb{V}_2^+$. Let $P_i := B \dot{s}_i B \sqcup B$ be a parabolic subgroup of G corresponding to s_i. Each $\mathbb{V}^+(i)$ is B-stable. Hence, it is P_i-stable. We have

$$\boldsymbol{\pi}(\mathbb{O}_i) = \overline{\mathsf{O}}_{s_i} = G(1 \times P_i)\mathsf{p}_1 \subset G/B \times G/B.$$

The product $(1 \times P_i)\mathsf{p}_1 \times \mathbb{V}^+(i)$ is a B-equivariant vector bundle. Here we have $G \cap (B \times P_i) = B$. Hence, we can induce it up to a G-equivariant vector bundle $\tilde{\mathbb{V}}(i)$ on $\boldsymbol{\pi}(\mathbb{O}_i)$. By means of the natural embedding of G-equivariant vector bundles

$$\boldsymbol{F} = G \times^B \mathbb{V}_2^+ \hookrightarrow G \times^B \mathbb{V}_2 \cong G \times \mathbb{V}_2,$$

we can naturally identify $\boldsymbol{\pi}^{-1}(\mathsf{p}_{s_i})$ with $\mathbb{V}^+(i)$. Since $\mathbb{V}^+(i)$ is P_i-stable, we conclude $\boldsymbol{\pi}^{-1}(\mathsf{p}_{s_i}) \cong \mathbb{V}^+(i)$ as P_i-modules. As a consequence, we conclude $\tilde{\mathbb{V}}(i) \cong \mathbb{O}_i$. Let $\breve{F}(i) := G \times^B (\mathbb{V}_2^+ / \mathbb{V}^+(i))$. It is a $\boldsymbol{G}$-equivariant quotient bundle of $\boldsymbol{F}$. The rank of $\breve{F}(i)$ is one $(1 \le i < n)$ or two $(i = n)$. Let $\breve{Z}_{\le s_i}$ be the image of $Z_{\le s_i}$ under the quotient map $\boldsymbol{F} \times \boldsymbol{F} \to \breve{F}(i) \times \breve{F}(i)$. We obtain the following commutative diagram:

$$\begin{tikzcd}
\boldsymbol{F} \arrow[d] & Z_{\le s_i} \arrow[l] \arrow[r] \arrow[d] & \boldsymbol{F} \arrow[d] \\
\breve{F}(i) & \breve{Z}_{\le s_i} \arrow[l] \arrow[r] & \breve{F}(i)
\end{tikzcd}$$

Here the above objects are smooth $\mathbb{V}^+(i)$-fibrations over the bottom objects. Therefore, it suffices to compute the convolution operation of the bottom line. We have $\breve{Z}_{\le s_i} = \overline{\mathsf{O}}_{s_i} \cup \triangle(\breve{F}(i))$, where $\triangle : \breve{F}(i) \hookrightarrow \breve{F}(i)^2$ is the diagonal embedding. Let $\breve{p}_j : \overline{\mathsf{O}}_{s_i} \to G/B$ $(j = 1, 2)$ be projections induced by the natural projections of $G/B \times G/B$. By construction, each $\breve{p}_j$ is a $\boldsymbol{G}$-equivariant $\mathbb{P}^1$-fibration. Let $\breve{\mathcal{L}}_\lambda$ be the pullback of $G \times^B \lambda^{-1}$ to $\breve{F}(i)$. We deduce

$$\tilde{T}_i \circ [\breve{\mathcal{L}}_\lambda] = \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* (\mathcal{O}_{\overline{\mathsf{O}}_{s_i}} \otimes^{\mathbb{L}} (\mathcal{O}_{\breve{F}(i)} \boxtimes \breve{\mathcal{L}}_\lambda)]$$

$$= \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* \breve{p}_2^* \imath_* (G \times^B \lambda^{-1})] = \left[G \times^B [\frac{e^{\lambda} - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}] \right],$$

where $\imath : G/B \hookrightarrow \breve{F}(i)$ is the zero section, and $[\frac{e^{\lambda}-e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}}] \in R(T) \cong R(B)$ is a virtual B-module. Here the ideal sheaf associated to $G/B \subset \breve{F}(i)$ represents $\mathbf{q}_2[\breve{\mathcal{L}}_{\alpha_i}]$ in $K^{\boldsymbol{G}}(\breve{F}(i))$ $(1 \le i < n)$ or corresponds to $\mathbf{q}_0\breve{\mathcal{L}}_{\epsilon_n} + \mathbf{q}_1\breve{\mathcal{L}}_{\epsilon_n} \subset \mathcal{O}_{\breve{F}(i)}$ $(i = n)$. In the latter case, divisors corresponding to $\mathbf{q}_0\breve{\mathcal{L}}_{\epsilon_n}$ and $\mathbf{q}_1\breve{\mathcal{L}}_{\epsilon_n}$ are normal crossing. Thus, we have $[\mathbf{q}_0\breve{\mathcal{L}}_{\epsilon_n} \cap \mathbf{q}_1\breve{\mathcal{L}}_{\epsilon_n}] = \mathbf{q}_0\mathbf{q}_1[\breve{\mathcal{L}}_{2\epsilon_n}]$. In particular, we deduce

$$[\mathbf{q}_0\breve{\mathcal{L}}_{\epsilon_n} + \mathbf{q}_1\breve{\mathcal{L}}_{\epsilon_n}] = \mathbf{q}_0[\breve{\mathcal{L}}_{\epsilon_n}] + \mathbf{q}_1[\breve{\mathcal{L}}_{\epsilon_n}] - \mathbf{q}_0\mathbf{q}_1[\breve{\mathcal{L}}_{2\epsilon_n}] \in K^{\boldsymbol{G}}(\breve{F}(n)).$$

Therefore, we conclude

$$\tilde{T}_i \circ e^{\lambda} = \begin{cases} (1-\mathbf{q}_2 e^{\alpha_i})\frac{e^{\lambda}-e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} & (1 \le i < n) \\ (1-\mathbf{q}_0 e^{\frac{\alpha_n}{2}})(1-\mathbf{q}_1 e^{\frac{\alpha_n}{2}})\frac{e^{\lambda}-e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} & (i = n) \end{cases}$$

as desired. □

The following representation of $\mathbb{H}$ is usually called the basic representation or the anti-spherical representation:

Theorem 2.7 (Basic representation c.f. [Mc03] 4.3.10)**.** *There is an injective $\mathcal{A}$-algebra homomorphism*

$$\varepsilon : \mathbb{H} \to \mathrm{End}_{\mathcal{A}}\mathcal{A}[T],$$

defined as $\varepsilon(e^{\lambda}) := e^{\lambda}\cdot$ $(\lambda \in X^*(T))$ and

$$\varepsilon(T_i)e^{\lambda} := \begin{cases} \frac{e^{\lambda}-e^{s_i\lambda}}{e^{\alpha_i}-1} - \mathbf{q}_2\frac{e^{\lambda}-e^{s_i\lambda+\alpha_i}}{e^{\alpha_i}-1} & (1 \le i < n) \\ \frac{e^{\lambda}-e^{s_n\lambda}}{e^{\alpha_n}-1} + \mathbf{q}_0\mathbf{q}_1\frac{e^{\lambda}-e^{s_n\lambda+\alpha_n}}{e^{\alpha_n}-1} - (\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n}\frac{e^{\lambda}-e^{s_n\lambda}}{e^{\alpha_n}-1} & (i = n) \end{cases}.$$

Theorem 2.8 (Exotic geometric realization of Hecke algebras)**.** *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z}),$$

as algebras.

Proof. Consider an assignment ϑ

$$\underline{e^{\lambda}} \mapsto \mathsf{e}^{\lambda}, \underline{T_i} \mapsto \begin{cases} \tilde{T}_i - (1-\mathsf{q}_2(\mathsf{e}^{\alpha_i}+1)) & (1 \le i < n) \\ \tilde{T}_i + (\mathsf{q}_0+\mathsf{q}_1)\mathsf{e}^{\epsilon_n} - (1+\mathsf{q}_0\mathsf{q}_1(\mathsf{e}^{\alpha_n}+1)) & (i = n) \end{cases}.$$

By means of the Thom isomorphism, the above assignment gives an action of

an element of the set $\{\underline{e}^{\lambda}\} \cup \{\underline{T_i}\}_{i=1}^{n}$ on $\mathcal{A}[T]$. We have

$$\begin{aligned}
\vartheta(\underline{e}^{\lambda})e^{\mu} =& e^{\lambda+\mu} \\
\vartheta(\underline{T_i})e^{\lambda} =& \left(\tilde{T}_i - (1 - \mathbf{q}_2(e^{\alpha_i}+1))\right) e^{\lambda} = (1-\mathbf{q}_2 e^{\alpha_i})\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - e^{\lambda} + \mathbf{q}_2(e^{\alpha_i}+1)e^{\lambda} \\
=& (\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^{\lambda} - e^{\lambda-\alpha_i}}{1-e^{-\alpha_i}}) - \mathbf{q}_2 e^{\alpha_i}(\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^{\lambda} - e^{\lambda-2\alpha_i}}{1-e^{-\alpha_i}}) = \varepsilon(T_i)e^{\lambda} \\
\vartheta(\underline{T_n})e^{\lambda} =& \left(\tilde{T}_n + (\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n} - (1+\mathbf{q}_0\mathbf{q}_1(e^{\alpha_n}+1))\right) e^{\lambda} \\
=& (1-\mathbf{q}_0 e^{\epsilon_n})(1-\mathbf{q}_1 e^{\epsilon_n})\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - e^{\lambda} + (\mathbf{q}_0+\mathbf{q}_1)e^{\lambda+\epsilon_n} - \mathbf{q}_0\mathbf{q}_1(e^{\alpha_n}+1)e^{\lambda} \\
=& (\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda} - e^{\lambda-\alpha_n}}{1-e^{-\alpha_n}}) + \mathbf{q}_0\mathbf{q}_1 e^{\alpha_n}(\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda} - e^{\lambda-2\alpha_n}}{1-e^{-\alpha_n}}) \\
& - (\mathbf{q}_0+\mathbf{q}_1)(\frac{e^{\lambda+\epsilon_n} - e^{s_n\lambda-\epsilon_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda+\epsilon_n} - e^{\lambda-\epsilon_n}}{1-e^{-\alpha_n}}) = \varepsilon(T_n)e^{\lambda}.
\end{aligned}$$

This identifies $\mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{F})$ with the basic representation of $\mathbb{H}$ via the correspondence $e^{\lambda} \mapsto \underline{e}^{\lambda}$ and $T_i \mapsto \underline{T_i}$. In particular, it gives an inclusion $\mathbb{H} \subset \mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z})$. Here we have $T_i \in \tilde{T}_i + R(\boldsymbol{T})$ for $1 \le i \le n$. It follows that $\mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z}) \subset \mathbb{H}$, which yields the result. □

Theorem 2.9 (Bernstein c.f. [CG97] 7.1.14 and [Mc03] 4.2.10)**.** *The center $Z(\mathbb{H})$ of $\mathbb{H}$ is naturally isomorphic to $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbf{G})$.* □

Corollary 2.10. *The center of $K^{\boldsymbol{G}}(\boldsymbol{Z})$ is $R(\boldsymbol{G})$.* □

For a semisimple element $a \in \boldsymbol{G}$, we define

$$\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H} \quad \text{(c.f. §1.4)}$$

and call it the specialized Hecke algebra.

Theorem 2.11. *Let $a \in \boldsymbol{G}$ be a semisimple element. We have an isomorphism*

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a)$$

as algebras.

Proof. This is a combination of [CG97] 6.2.3 and 5.10.11. (See also [CG97] 8.1.6.) □

Convention 2.12. Let $a = (s, \vec{q}) \in \boldsymbol{G}$ be a pre-admissible element. We define Z_+^a to be the image of $\boldsymbol{Z}^a$ under the natural projection defined by

$$\boldsymbol{Z} \ni (g_1B, g_2B, X_0, X_1, X_2) \mapsto (g_1B, g_2B, X_0 + X_1, X_2) \in Z.$$

Let F_+^a be the image of Z_+^a via the first (or the second) projection. Let μ_+^a be the restriction of μ to F_+^a. We denote its image by $\mathfrak{N}_+^a$. By the assumption $q_0 \neq q_1$, we have $F_+^a \cong \boldsymbol{F}^a$, $Z_+^a \cong \boldsymbol{Z}^a$, and $\mathfrak{N}_+^a \cong \mathfrak{N}_2^a$.

Corollary 2.13. *Keep the setting of Convention 2.12. We have an isomorphism*

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z_+^a)$$

as algebras. □

For the later use, let us introduce our last class of parameter here.

Definition 2.14 (Regular parameters)**.** A pre-admissible parameter (a, X) is called regular iff there exists a direct factor $A[d] \subset (\mu_+^a)_* \mathbb{C}_{F_+^a}$, where A is a simple $\boldsymbol{G}(a)$-equivariant perverse sheaf on $\mathfrak{N}_+^a$ such that $\mathrm{supp} A = \overline{\boldsymbol{G}(a)X}$ and d is an integer.
We denote by $\mathfrak{R}_a$ the set of $\boldsymbol{G}(a)$-conjugacy classes of regular pre-admissible parameters of the form (a, X) $(X \in \mathfrak{N}_+^a)$.

3 Clan decomposition

We work under the same setting as in §2.

Definition 3.1 (Clans)**.** Let $a = (s, \vec{q})$ be a pre-admissible element such that $s \in T$. Let $q_2 = e^{r_2}$. We put $\Gamma := r_2\mathbb{Z} + \Gamma_0$. A clan associated to a is a maximal subset $\mathbf{c} \subset [1, n]$ with the following property: For each two elements $i, j \in \mathbf{c}$, there exists a sequence $i = i_0, i_1, \ldots, i_m = j$ (in $\mathbf{c}$) such that

$$\{\log_{i_k}(s) \pm \log_{i_{k+1}}(s)\} \cap \{\pm r_2 + \Gamma_0, \Gamma_0\} \neq \emptyset \quad \text{for each } 0 \le k < m.$$

We have a disjoint decomposition

$$[1, n] = \bigsqcup_{\mathbf{c} \in \mathcal{C}_a} \mathbf{c},$$

where each $\mathbf{c}$ is a clan associated to a and $\mathcal{C}_a$ is the set of clans associated to a. For a clan $\mathbf{c}$, we put $n^{\mathbf{c}} := \# \mathbf{c}$.

We assume the setting of Definition 3.1 in the rest of this section unless stated otherwise. At the level of Lie algebras, we have a decomposition

$$\mathfrak{g}(s) := \mathfrak{t} \oplus \bigoplus_{\substack{i<j, \sigma_1, \sigma_2 \in \{\pm 1\}, \\ \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in [1,n], \sigma \in \{\pm 1\}, \\ 2\log_i(s) \equiv 0}} \mathfrak{g}(s)[\sigma 2 \epsilon_i],$$

where $\equiv$ means modulo Γ_0. For each $\mathbf{c} \in \mathcal{C}_a$, we define a Lie algebra $\mathfrak{g}(s)_{\mathbf{c}}$ as the Lie subalgebra of $\mathfrak{g}(s)$ defined as

$$\bigoplus_{i \in \mathbf{c}} \mathbb{C}\epsilon_i \oplus \bigoplus_{\substack{i<j \in \mathbf{c}, \sigma_1, \sigma_2 \in \{\pm 1\}, \\ \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in \mathbf{c}, \sigma \in \{\pm 1\}, \\ 2\log_i(s) \equiv 0}} \mathfrak{g}(s)[\sigma 2 \epsilon_i],$$

where $\equiv$ means modulo Γ_0. Moreover, we have

$$\mathfrak{g}(s) = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathfrak{g}(s)_{\mathbf{c}}. \tag{3.1}$$

In particular, we have $[\mathfrak{g}(s)_{\mathbf{c}}, \mathfrak{g}(s)_{\mathbf{c}'}] = 0$ unless $\mathbf{c} = \mathbf{c}'$. Let $G(s)_{\mathbf{c}}$ be the connected subgroup of $G(s)$ which has $\mathfrak{g}(s)_{\mathbf{c}}$ as its Lie algebra.

The following theorem is a consequence of Steinberg's centralizer theorem and the Borel-de Siebenthal theorem, for which we present a proof for the reference purpose.

Theorem 3.2 (Centralizer theorem for symplectic groups)**.** *Let $A \subset T$ be an algebraic subgroup. Then, the group $G(A)$ is connected.*

Proof. By a Lie algebra calculation, the group $G(A)^\circ$ is generated by T and unipotent one-parameter subgroups U_α ($\alpha \in R$) such that $\alpha(A) = \{1\}$. By a repeated use of the Borel-de Siebenthal theorem [BS49], the root system of $G(A)^\circ$ is the product of the standard presentations of the root systems of

$$GL(m, \mathbb{C}), SL(2, \mathbb{C}), \text{ or } Sp(2m, \mathbb{C}). \tag{3.2}$$

In particular, the derived group of $G(A)^\circ$ must be simply connected. Now we prove the theorem by induction on the cardinality k of a generating set of A. (Notice that the word "generating" means the Zariski closure of the group generated by a given subset of T is A. Hence, we can assume the finiteness of the cardinality of a such set.) The case $k = 1$ is an immediate consequence of Steinberg's centralizer theorem (c.f. [Ca85] 3.5.6). If the assertion is true for smaller k, then it suffices to consider the centralizer of a semi-simple element in a group listed at (3.2). This is again connected by Steinberg's centralizer theorem. Therefore, the induction proceeds and we obtain the result. □

Lemma 3.3. *We have $G(s) = \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$.*

Proof. By (3.1), it is clear that $\prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$ is equal to the identity component of $G(s)$. Since G is a simply connected semi-simple group, it follows that $G(s)$ is connected by Theorem 3.2. In particular, we have $G(s) \subset \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$ as desired. □

We denote $B \cap G(s)_{\mathbf{c}}$ and ${}^w B \cap G(s)_{\mathbf{c}}$ by $B(s)_{\mathbf{c}}$ and ${}^w B(s)_{\mathbf{c}}$, respectively.

Convention 3.4. We denote by $\mathbb{V}^a$ the image of $\mathbb{V}_2^a$ to $\mathbb{V}$ via the map

$$\mathbb{V}_2 \ni (X_0 \oplus X_1 \oplus X_2) \mapsto ((X_0 + X_1) \oplus X_2) \in \mathbb{V}.$$

Since $q_0 \neq q_1$, we have $\mathbb{V}^a \cong \mathbb{V}_2^a$.

For each $\mathbf{c} \in \mathcal{C}_a$, we define

$$\mathbb{V}_{\mathbf{c}}^a := \sum_{i,j \in \mathbf{c}, \sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}} \mathbb{V}^a[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \mathbb{V}^a[\sigma_3 \epsilon_i].$$

It is clear that $\mathbb{V}^a = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{V}_{\mathbf{c}}^a$. By the comparison of weights, the $\mathfrak{g}(s)_{\mathbf{c}}$-action on $\mathbb{V}_{\mathbf{c}'}^a$ is trivial unless $\mathbf{c} = \mathbf{c}'$.

Remark 3.5. Since $\mathbf{c}$ is not an integer and we do not use $\mathbb{V}_\ell$ in the rest of this paper (except for §7), we use the notation $\mathbb{V}_{\mathbf{c}}^a$. The author hopes the reader not to confuse $\mathbb{V}_{\mathbf{c}}^a$ with $(\mathbb{V}_\ell)^a$.

Lemma 3.6. *Let $\mathbb{O} \subset \mathfrak{N}_+^a$ be a $\boldsymbol{G}(a)$-orbit. Let $\mathbb{O}_{\mathbf{c}}$ denote the image of $\mathbb{O}$ under the natural projection $\mathbb{V}^a \to \mathbb{V}_{\mathbf{c}}^a$. Then, we have a product decomposition $\mathbb{O} = \oplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{O}_{\mathbf{c}}$.*

Proof. Let $X \in \mathbb{V}^a$. There exists a family $\{X_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}_a}$ ($X_{\mathbf{c}} \in \mathbb{V}_{\mathbf{c}}^a$) such that $X = \sum_{\mathbf{c} \in \mathcal{C}_a} X_{\mathbf{c}}$. We have $G(s)X = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}} X_{\mathbf{c}}$. For each of $i = 0, 1$, the clan $\mathbf{c} \in \mathcal{C}_a$ such that $(V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}}) \neq \{0\}$ is at most one since clans are

determined by the s-eigenvalues of V_1. Let $\mathbf{c}^i$ $(i = 1, 2)$ be the unique clan such that $(V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}^i}) \neq \{0\}$. Let $\mathbb{G}_{\mathbf{c}}$ be the product of scalar multiplications of $V_1^{(s,q_i)}$ such that $V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}}^a \neq \{0\}$. Since the set of a-fixed points of a conic variety in $\mathbb{V}$ is conic, we have $(G(s)_{\mathbf{c}} \times (\mathbb{C}^\times)^3)X_{\mathbf{c}} = (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}})X_{\mathbf{c}}$. We have $\prod_{\mathbf{c} \in \mathcal{C}_a} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) \subset \boldsymbol{G}(a)$. It follows that

$$\boldsymbol{G}(a)X = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \boldsymbol{G}(a)X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}})X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{O}_{\mathbf{c}}$$

as desired. □

For each $w \in W$, we define

$$F_+^a(w) := G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^a).$$

Similarly, we define

$$F_+^a(w, \mathbf{c}) := G(s)_{\mathbf{c}} \times^{{}^w B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a)$$

for each $\mathbf{c} \in \mathcal{C}_a$.

Lemma 3.7. *We have $F_+^a = \cup_{w \in W} F_+^a(w)$.*

Proof. The set of a-fixed points of G/B is a disjoint union of flag varieties of $G(s)$. It follows that each point of F_+^a is $G(s)$-conjugate to a point in the fiber over a T-fixed point of G/B. □

The local structures of these connected components are as follows.

Lemma 3.8. *For each $w \in W$, we have*

$$F_+^a(w) \cong \prod_{\mathbf{c} \in \mathcal{C}_a} F_+^a(w, \mathbf{c}).$$

Proof. The set $\mathbb{V}_{\mathbf{c}}^a$ is T-stable for each $\mathbf{c} \in \mathcal{C}_a$. Hence, we have

$$F_+^a(w) = G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^a) \cong G(s) \times^{{}^w B(s)} (\bigoplus_{\mathbf{c} \in \mathcal{C}_a} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a)).$$

Since we have $G(s)/B(s) \cong \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}/B(s)_{\mathbf{c}}$, we deduce

$$G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a) \cong \prod_{\mathbf{c}' \in \mathcal{C}_a} G(s)_{\mathbf{c}'} \times^{{}^w B(s)_{\mathbf{c}'}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a \cap \mathbb{V}_{\mathbf{c}'}^a).$$

Here the RHS is isomorphic to

$$F_+^a(w, \mathbf{c}) \times \prod_{\mathbf{c} \neq \mathbf{c}'} G(s)_{\mathbf{c}'}/{}^w B(s)_{\mathbf{c}'}.$$

Gathering these information yields the result. □

We define a map ${}^w\mu^a_{\mathbf{c}}$ by

$$ {}^w\mu^a_{\mathbf{c}} : F^a_+(w,\mathbf{c}) = G(s)_{\mathbf{c}} \times^{{}^w B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^a_{\mathbf{c}}) \longrightarrow \mathbb{V}^a_{\mathbf{c}}. $$

We put $G_{\mathbf{c}} := Sp(2n^{\mathbf{c}})$ and $s_{\mathbf{c}} := \exp(\sum_{i\in\mathbf{c}}(\log_i(s))\epsilon_i) \in T$. We have embeddings

$$ s = \prod_{\mathbf{c}\in\mathcal{C}_a} s_{\mathbf{c}} \in \prod_{\mathbf{c}\in\mathcal{C}_a} Sp(2n^{\mathbf{c}}) \subset Sp(2n), $$

induced by the following identifications:

$$ \mathfrak{g}(s)_{\mathbf{c}} = \mathfrak{g}_{\mathbf{c}}(s_{\mathbf{c}}) \subset \left(\bigoplus_{i\in\mathbf{c}} \mathbb{C}\epsilon_i\right) \oplus \bigoplus_{\substack{\alpha=\sigma_1\epsilon_i+\sigma_2\epsilon_j\neq 0\\ \sigma_1,\sigma_2\in\{\pm1\},\, i,j\in\mathbf{c}}} \mathfrak{g}[\alpha] = \mathfrak{g}_{\mathbf{c}}. \tag{3.3} $$

Note that we have $G(s)_{\mathbf{c}} = G_{\mathbf{c}}(s_{\mathbf{c}}) \subsetneq G_{\mathbf{c}}$ in general.

Let $\mathbb{V}(\mathbf{c})$ be the 1-exotic representation of $G_{\mathbf{c}}$. We have a natural embedding $\mathbb{V}^a_{\mathbf{c}} \subset \mathbb{V}(\mathbf{c})$ of $G(s)_{\mathbf{c}}$-modules. (The $G(s)_{\mathbf{c}}$-module structure on the RHS is given by the restriction of the $G_{\mathbf{c}}$-action.)

Let $\nu = (a, X)$ be a pre-admissible parameter. We have a family of pre-admissible parameters $\nu_{\mathbf{c}} := (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}})$ of $G_{\mathbf{c}}$'s such that $s = \prod_{\mathbf{c}} s_{\mathbf{c}}$, $X = \oplus_{\mathbf{c}} X_{\mathbf{c}}$. We denote

$$ \nu = \prod_{\mathbf{c}\in\mathcal{C}_a} \nu_{\mathbf{c}} $$

and call it the clan decomposition of ν. Let $W_a := \prod_{\mathbf{c}\in\mathcal{C}_a} N_{G_{\mathbf{c}}}(T)/T$. By Lemma 3.8, we conclude that

$$ \bigcup_{w\in W_a} F^a_+(w) \subset F^a_+ \tag{3.4} $$

is the product of the F^a_+'s obtained by replacing the pair (G,ν) by $(G_{\mathbf{c}}, \nu_{\mathbf{c}})$ for all $\mathbf{c} \in \mathcal{C}_a$.

Proposition 3.9 (Clan decomposition of μ^a)**.** *For each $w \in W$, we have*

$$ \mu^a_+|_{F^a_+(w)} \cong \prod_{\mathbf{c}\in\mathcal{C}_a} {}^w\mu^a_{\mathbf{c}}. $$

In particular, every irreducible direct summand A of $(\mu^a_+)_*\mathbb{C}_{F^a_+}$ is written as an external product of $G(s)$-equivariant sheaves appearing in $({}^w\mu^a_{\mathbf{c}})_*\mathbb{C}_{F^a_+(w,\mathbf{c})}$ (up to degree shift).

Proof. The first assertion follows from the combination of Lemma 3.6, Lemma 3.8, and the definition of ${}^w\mu^a_{\mathbf{c}}$. We have $\mathbb{C}_{F^a_+} = \bigoplus_{F^a_+(w)\subset F^a_+} \mathbb{C}_{F^a_+(w)}$. A direct summand of $(\mu^a_+)_*\mathbb{C}_{F^a_+}$ is a direct summand of $(\mu^a_+)_*\mathbb{C}_{F^a_+(w)}$ for some $w \in W$. Since

$$ (\mu^a_+)_*\mathbb{C}_{F^a_+(w)} \cong \boxtimes_{\mathbf{c}} ({}^w\mu^a_{\mathbf{c}})_*\mathbb{C}_{F^a_+(w,\mathbf{c})}, $$

the second assertion follows. □

Corollary 3.10. *Let $\nu = (a, X)$ be a pre-admissible parameter. Then, it is regular if and only if $\nu_{\mathbf{c}}$ is a regular pre-admissible parameter of $G_{\mathbf{c}}$ for every $\mathbf{c} \in \mathcal{C}_a$.*

Proof. Let $W_0 := N_{G(s)}(T)/T \subset W$. We have a natural inclusion $W_0 \subset W_a$. Here we have
$$\mu_+^a = \bigsqcup_{w \in W/W_0} \mu_+^a|_{F_+^a(w)},$$
where we regard $W/W_0 \subset W$ by taking some representative. For each $w \in W$, there exists $v \in W_a$ such that ${}^w\mathbb{V}^+ \cap \mathbb{V}^a = {}^v\mathbb{V}^+ \cap \mathbb{V}^a \subset \mathbb{V}^a$. Moreover, we can choose v so that ${}^wB(s)_{\mathbf{c}} = {}^vB(s)_{\mathbf{c}}$ holds for each $\mathbf{c} \in \mathcal{C}_a$. As a consequence, all $F_+^a(w)$ are isomorphic to one of $F_+^a(w)$ ($w \in W_a$) as $\boldsymbol{G}(a)$-varieties, together with maps $\mu_+^a|_{F_+^a(w)}$ to $\mathbb{V}^a$. Therefore, ν is regular if and only if an intersection cohomology complex with its support $\overline{\boldsymbol{G}(a)\nu}$ (with degree shift) appears in $(\mu_+^a)_*\mathbb{C}_{F_+^a(w)}$ for some $w \in W_a$. Hence, Proposition 3.9 implies the result. $\square$

Corollary 3.10 reduces the analysis of the decomposition pattern of $(\mu_+^a)_*\mathbb{C}_{F_+^a}$ into the case that ν has a unique clan.

4 On stabilizers of exotic nilpotent orbits

We retain the setting of §2.

Lemma 4.1. *Let H be a connected linear algebraic group and let $\mathcal{X}$ be a variety with H-action. Let $H = H_rH_u$ be a Levi decomposition of H with H_r its reductive part. If $\mathsf{Stab}_{H_r}x$ is connected for $x \in \mathcal{X}$, then so is $\mathsf{Stab}_H x$.*

Proof. Assume to the contrary to deduce contradiction. Let $h \in \mathsf{Stab}_H x$ be an element which is not in the identity component. Let $h = h_rh_u \in H_rH_u$ be its Levi decomposition. For some $k > 1$, we have $h^k \in (\mathsf{Stab}_H x)^\circ$. This implies the existence of $g \in (\mathsf{Stab}_H x)^\circ$ which satisfies $h^k = g^k$. Let $g = g_rg_u$ be the Levi decomposition. We have $H_u \lhd H$, which claims $h_r^k = g_r^k$. Replacing h by $g^{-1}h$, we further assume $h_r^k = 1$. Here we have $h^k \in (\mathsf{Stab}_{H_u} x)^\circ = \mathsf{Stab}_{H_u} x$. Put $u := h^k \in \mathsf{Stab}_{H_u} x$. Let U denote the group given as the Zariski closure of the group generated by h. We have its connected component decomposition $U = U_0 \sqcup U_1 \sqcup U_2 \sqcup \cdots$, where $1, u \in U_0$ and $h \in U_1$. Since h_r is of finite order, U_0 is unipotent and each of U_i is a homogeneous U_0-space. Let $U_0^{(m)}$ be the m-th lower central subgroup of U_0. For each m, the adjoint h-action preserves $U_0^{(m)}$. It follows that if $u \in U_0^{(m)}$ for some m, then we have $(hu_m)^k \in U_0^{(m)}$ for each $u_m \in U_0^{(m)}$. Moreover, we have $huh^{-1} \equiv u \mod U_0^{(m+1)}$ by $u = h^k$. Since $U_0^{(m)}/U_0^{(m+1)}$ is abelian, we deduce
$$\{(hu_m)^k; u_m \in U_0^{(m)}\}/U_0^{(m+1)} = \{\bar{u}_m \in U_0^{(m)}/U_0^{(m+1)}; h\bar{u}_mh^{-1} = \bar{u}_m\} \subset U_0^{(m)}/U_0^{(m+1)}$$
for each m. The second term contains 1 mod $U_0^{(m+1)}$. We have $U_0^{(m)} = \{1\}$ for $m \gg 0$. Hence, we can change h if necessary to assume $h^k = 1$, which implies that h is semisimple. Therefore, h belongs to $\mathsf{Stab}_{H_r} x$. An element of finite order is always semisimple, hence its unipotent part is trivial. Thus, we have $h_u = 1$ if $h^r = 1$. Therefore, we have contradiction and the result follows. $\square$

Theorem 4.2 (Igusa [Ig73] Lemma 8, Springer [Sp07])**.** *Let $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ be a partition of n. We regard it as a 0-marked partition. Then, the reductive part of $\mathsf{Stab}_G\mathbf{v}_\lambda$ is*
$$L_\lambda := Sp(2n_1, \mathbb{C}) \times Sp(2n_2, \mathbb{C}) \times \cdots,$$

where the sequence $(n_1, n_2, \ldots)$ are the number of λ_i's which share the same value. Moreover, we have

$$\mathsf{Res}^G_{L_\lambda} V_1 = \bigoplus_{i \geq 1} V(i)^{\oplus \lambda_i}, \tag{4.1}$$

where $V(i)$ is the vector representation of $Sp(2n_i)$ with trivial actions of $Sp(2n_j)$ $(j \neq i)$. □

Remark 4.3. Igusa's result is not as precise as Theorem 4.2. But we can deduce from its proof without difficulty. Springer [Sp07] contains more precise statement.

Corollary 4.4. *Keep the setting of Theorem 4.2. Then, we can choose a maximal torus of L_λ inside T.*

Proof. Let $\sigma = (\mathbf{J}, \emptyset)$ be the 0-marked partition corresponding to λ. By Lemma 1.18, we have $\mathbb{C}^\times \subset \mathsf{Stab}_{T_J} \mathbf{v}^J_\sigma$ for each $J \in \mathbf{J}$. It follows that $L_\lambda \cap T$ contains a torus of dimension $(\sum_{i \geq 0} n_i)$, which implies the result. □

Proposition 4.5. *Let $X \in \mathfrak{N}_2$. Then, $\mathsf{Stab}_G X$ is connected.*

Proof. Let $X = (X_0 \oplus X_1) \oplus \mathbf{v}_{2,\lambda}$, where λ is a partition of n regarded as a 0-marked partition. It suffices to show that the action of $\mathsf{Stab}_G \mathbf{v}_{2,\lambda}$ on $(X_0 \oplus X_1)$ has connected stabilizer. Let $L_\lambda U_\lambda$ be the Levi decomposition of $\mathsf{Stab}_G \mathbf{v}_{2,\lambda}$. By Lemma 4.1, it is sufficient to show that the stabilizer of L_λ on $(X_0 \oplus X_1)$ is connected. By Theorem 4.2, it suffices to prove that the G-stabilizer of finite set of elements in V_1 is connected. By a repeated use of Lemma 4.1, it suffices to prove that the G-stabilizer of one element in V_1 has $Sp(2n-2)$ as its (reductive) Levi factor. We denote the element $v \in V_1$ and fix a symplectic form on V_1 which is preserved by G. Then, it is easy to see that $\mathsf{Stab}_G v$ preserves $\mathbb{C}v$ and the compliment space $v^\perp$ of V_1 with respect to the symplectic form. Thus, its Levi component is given as a subgroup of

$$\mathbb{C}^\times \times Sp(2n-2) = \left(\mathbb{C}^\times \times GL(2n-2, \mathbb{C}) \times \mathbb{C}^\times\right) \cap Sp(2n) \subset GL(V_1),$$

which fixes v. (Here the middle group is the Levi component of $GL(V_1)$ which preserves a partial flag $\{0\} \subset \mathbb{C}v \subset v^\perp \subset V_1$.) Therefore, it is $Sp(2n-2)$ as desired. □

Remark 4.6. Springer [Sp07] contains an explicit description of the G-stabilizer of each strict normal form. As is seen easily from the proof of Proposition 4.5, it is not hard to write down the G-stabilizer of a point of $\mathfrak{N}_2$ assuming [Sp07].

Corollary 4.7 (of the proof of Proposition 4.5)**.** *For each $X \in \mathfrak{N}_2$, the reductive part of $\mathsf{Stab}_G X$ is a product of symplectic groups.* □

Theorem 4.8 (Refined form of Theorem 1.17)**.** *Let $\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2) \in \boldsymbol{T} \times \mathfrak{N}$ be a pre-admissible parameter which is admissible or $a = a_0$. Then, we have a clan decomposition*

$$\nu = \prod_{\mathbf{c} \in \mathcal{C}_a} \nu^{\mathbf{c}} = \prod_{\mathbf{c} \in \mathcal{C}_a} (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}})$$

with the following properties:

- *Each $\nu_{\mathbf{c}}$ is an admissible parameter;*
- *There exists $g \in G$ such that:*

$$gsg^{-1} \in T \text{ and each } gX_{\mathbf{c}} \text{ is a strict normal form.}$$

Proof. If $a = a_0$, then we have $\mathfrak{N}_2^a \cong \mathfrak{N}$. Hence, the result reduces to Proposition 1.16 1).

Thus, we assume that a is admissible. Since the admissibility condition depends only on the configuration of $\vec{q}$, the clan decomposition preserves admissibility. Hence, it suffices to prove the case that $\mathbf{c} = [1, n]$ is the unique clan of $\mathcal{C}_a$. Then, two distinct eigenvalues t_1, t_2 of s on V_1 satisfies

$$t_1 t_2 \text{ or } t_1/t_2 = q_2^m, \text{ where } |m| < n.$$

It follows that at least one of q_0 or q_1 does not appear as a s-eigenvalue of V_1 by the admissibility condition. Therefore, we can assume $(s - q_1)X_1 = 0$ by swapping the roles of q_0 and q_1 if necessary.

Let us take G-conjugate to assume that $X = \mathbf{v}_\sigma$ for a strict marked partition $\sigma = (\mathbf{J}, \vec{\delta})$. By the description of the G-stabilizer of $\mathbf{v}_\sigma$, we deduce that we can choose a maximal torus of $\mathsf{Stab}_G \mathbf{v}_\sigma$ inside of T. By Lemma 1.18 and the fact $\mathbf{v}_\sigma$ is a strict normal form, we deduce that a (possibly disconnected) maximal torus of $\mathsf{Stab}_{\boldsymbol{G}} \mathbf{v}_\sigma$ is taken inside $\boldsymbol{T}$. Therefore, we conclude that $(a, \mathbf{v}_\sigma)$ is a strict normal form after taking conjugate of a by the $\mathsf{Stab}_{\boldsymbol{G}} \mathbf{v}_\sigma$-action (or the $\mathsf{Stab}_G \mathbf{v}_\sigma$-action). ☐

Corollary 4.9. *Let $a = (s, q_0, q_1, q_2) \in \boldsymbol{T}$ be an admissible element. If $\mathcal{C}_a$ consists of a unique clan $[1, n]$, then we have either $V_1^{(s,q_0)} = \{0\}$ or $V_1^{(s,q_1)} = \{0\}$.*

Proof. See the second paragraph of the proof of Proposition 4.8. ☐

Theorem 4.10. *Let $(a, X) = (s, \vec{q}, X)$ be a pre-admissible parameter. Then, $\mathsf{Stab}_{G(s)} X$ is connected.*

Proof. The group $\mathsf{Stab}_{G(s)} X$ consists of elements of $\mathsf{Stab}_G X$ which commute with s inside G. Moreover, this is equal to $(G \cap (\mathsf{Stab}_{\mathbf{G}} X)(a))$. Let $\mathbf{L}$ be the Levi part of $\mathsf{Stab}_{\mathbf{G}} X$. By Lemma 4.1, the desired component group is the same as that of $(G \cap \mathbf{L}(a))$. By Corollary 4.7, we deduce that $\mathbf{L}$ is a product of symplectic groups and a torus $T^\perp$ which injects into $(\mathbb{C}^\times)^3$ via the second projection $\mathbf{G} = G \times (\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^3$.

We take G-conjugation if necessary to assume that X_2 is a strict 0-normal form corresponding to a partition λ of n and $s \in T$ by Theorem 4.8. Then, the semi-simple groups contributing $\mathbf{L}$ are direct factors of the subgroup of the group $L = L_\lambda$ borrowed from Theorem 4.2 which fix the both of X_0 and X_1. The T-action on V_1 is compatible with the restriction of (4.1). It follows that we have a sequence of semi-simple elements $s_1, s_2, \ldots,$ in L such that

$$Z_L(s) = \{g \in L; gs = sg\} = \bigcap_{j \geq 1} L(s_j).$$

Let A be the Zariski closure of the group generated by $s_1, \ldots$ in L. The condition that an element of L fixes X_0 or X_1 can be translated into a condition that a

collection of vectors $\{X_0^1, X_0^2, \ldots,\}$ or $\{X_1^1, X_1^2, \ldots,\}$ of $\oplus_i V(i)$ obtained from X_0 or X_1 by (4.1) is fixed, respectively. We put $S \subset \oplus_i V(i)$ to be the A-span of all the vectors in $\{X_0^1, \ldots\} \cup \{X_1^1, \ldots\}$. The condition that an element of $L(A)$ fixes X_0 and X_1 is the same as fixing each element of S. Here the subgroup L' of L which fixes S is a product of (probably smaller) symplectic groups as in the proof of Proposition 4.5. Moreover, the subgroup of $L(A)$ which fixes S is isomorphic $L'(A')$ for some torus $A' \subset L'$ obtained as the Zariski closure of elements of L' which acts as the same as $s_1, \ldots$ to $S^{\perp}/S$. (Here $S^{\perp}$ is the orthogonal complement of S with respect to the G-invariant symplectic form on V_1.)

Therefore, we deduce that $\mathsf{Stab}_{L(A)}(X_0 \oplus X_1)$ is written as a product of the centralizer of some subgroups of maximal torus in symplectic groups. Since each of such groups are connected by Theorem 3.2, we conclude the result. $\square$

5 Semisimple elements attached to $G \backslash \mathfrak{N}_1$

We keep the setting of the previous section.

Let $\sigma := (\mathbf{J}, \vec{\delta})$ be a strict marked partition. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be the partition of n corresponding to $\mathbf{J} = \{J_1, J_2, \ldots\}$.

We fix a sequence of positive real numbers $\gamma_0, \gamma_1, \ldots, \gamma_n > (n+1)\gamma$ such that

$$\{2\gamma_i, 2\gamma_j, \gamma_i + \gamma_j, \gamma_i - \gamma_j\} \cap (\Gamma + \mathbb{Z}\gamma) = \emptyset \tag{5.1}$$

holds for every pair of distinct numbers (i, j) in $[0, n]$.

Remark 5.1. Our choice of $\{\gamma_k\}_k$ and γ are possible since $\mathbb{C}$ is an extension of the field $\mathbb{Q}(q_2, \sqrt{-1}, \pi)$ with infinite transcendental degree.

We define a semi-simple element $s_\sigma \in T$ as follows:

- If $\delta_1|_{J_k} \equiv 0$, then we set $\log_j s_\sigma = \gamma_k - j\gamma$ for each $j \in J_k$;
- If $\delta_1(j_0) = 1$ for $j_0 \in J_k$, then we set $\log_j s_\sigma = \gamma_0 - (j - j_0)\gamma$ for each $j \in J_k$.

By the definition of strict marked partitions, the choice of j_0 is unique for each $J \in \mathbf{J}$. Hence, s_σ is uniquely determined. We put $a_\sigma := (s_\sigma, e^{\gamma_0}, 1, e^{\gamma}) \in \boldsymbol{T}$.

Lemma 5.2. *In the above setting, we have $a_\sigma \mathbf{v}_\sigma = \mathbf{v}_\sigma$.*

Proof. It suffices to prove $(s_\sigma, e^{\gamma_0}, 1, e^{\gamma})\mathbf{v}_\sigma^J = \mathbf{v}_\sigma^J$ for each $J \in \mathbf{J}$. Let $J = J_k$. Then, $\mathbf{v}_{2,\sigma}^{J_k}$ is a sum of $y_{i,i+1}$ for $i, i+1 \in J_k$, which has s_σ-eigenvalue

$$e^{(\gamma_k - i\gamma - (\gamma_k - (i+1)\gamma))} = e^{\gamma} \text{ or } e^{(\gamma_0 - (i - j_0)\gamma - (\gamma_0 - (i+1-j_0)\gamma))} = e^{\gamma},$$

where the latter case occurs only if $\delta(j_0) = 1$ for some $j_0 \in J_k$. Hence, we have $s_\sigma \mathbf{v}_{2,\sigma}^{J_k} = e^{\gamma}\mathbf{v}_{2,\sigma}^{J_k}$. Moreover, we have $s_\sigma x_i = e^{\gamma_0} x_i$ if $\delta_1(i) = 1$. In particular, we have $s_\sigma \mathbf{v}_{1,\sigma}^{J_k} = e^{\gamma_0}\mathbf{v}_{1,\sigma}^{J_k}$. These calculations imply the desired result. $\square$

Fix a real number $\mathsf{r} > 0$. We define $D_\sigma \in T$ to be

$$\log_i D_\sigma = \begin{cases} 0 & (\log_i s_\sigma \notin \gamma_0 + \Gamma) \\ -\mathsf{r}(\underline{\#} J_k) & (i \in J_k \ni \exists j_0, \delta_1(j_0) = 1) \end{cases}$$

Consider a parabolic subgroup P_σ of $G(s_\sigma)$:

$$P_\sigma := \{g \in G(s_\sigma); \lim_{N \to \infty} \mathrm{Ad}(D_\sigma^N) g \in G(s_\sigma)\}.$$

It is well-known that P_σ is a parabolic subgroup of $G(s_\sigma)$. Let w_σ be the shortest element of W such that

$$\langle w_\sigma R^+, D_\sigma \rangle \leq 1.$$

It is straight-forward to see

$$^{w_\sigma} B \cap G(s_\sigma) \subset P_\sigma.$$

Lemma 5.3. *For a strict marked partition σ, we have $\mathbf{v}_\sigma \in \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+$.*

Proof. By Lemma 5.2, it suffices to prove $\mathbf{v}_\sigma \in {}^{w_\sigma}\mathbb{V}^+$. The definition of w_σ implies that

1. $x_i \in {}^{w_\sigma}\mathbb{V}^+$ if and only if **a)** $D_\sigma(\epsilon_i) < 1$ or **b)** $D_\sigma(\epsilon_i) = 1$ and $i > 0$;
2. $y_{ij} \in {}^{w_\sigma}\mathbb{V}^+$ if and only if **a)** $D_\sigma(\epsilon_i - \epsilon_j) < 1$ or **b)** $D_\sigma(\epsilon_i - \epsilon_j) = 1$ and $\epsilon_i - \epsilon_j \in R^+$.

Since $\mathbf{v}_{1,\sigma}$ is sum of x_i with D_σ-eigenvalue < 1, we have $\mathbf{v}_{1,\sigma} \in {}^{w_\sigma}\mathbb{V}^+$. The vector $\mathbf{v}_{2,\sigma}$ have D_σ-eigenvalue 1. By construction, a strict normal form is contained in $\mathbb{V}^+$. Therefore, we conclude $\mathbf{v}_{2,\sigma} \in {}^{w_\sigma}\mathbb{V}^+$, which completes the proof. □

Proposition 5.4. *Let $\sigma = (\mathbf{J}, \vec{\delta})$ be a strict marked partition. Then, we have an inclusion*

$$P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+,$$

which is dense open.

Before giving the proof of Proposition 5.4, we count the set of weights we concern in its proof:

Lemma 5.5. *Keep the setting of Proposition 5.4. Then, the set $\Psi(\mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+)$ is given by the following list:*

1. *$\epsilon_i - \epsilon_{i+1}$ for each $i, i+1 \in J \in \mathbf{J}$;*
2. *$\epsilon_{i+j_0} - \epsilon_{i+j_1+1}$ if the following conditions hold:*
   - *$i + j_0, j_0 \in J_k$, and $i + j_1 + 1, j_1 \in J_{k'}$ for some k, k';*
   - *$\delta_1(j_0) = 1 = \delta_1(j_1)$, and $\underline{\#} J_k > \underline{\#} J_{k'}$;*
3. *ϵ_{j_0} for each $j_0 \in J_k$ such that $\delta(j_0) = 1$.*

Proof. In this proof, we assume all integer index (which are *a priori* not necessarily positive) to be positive. By the choice of the sequence $\{\gamma_k\}_k$, we have

$$|\langle s_\sigma, \epsilon_j + \epsilon_{j'} \rangle| \geq e^{-2n\gamma} \min\{e^{\gamma_k + \gamma_{k'}}; k, k' \in [1, n]\} > e^\gamma$$

for each j, j'. It follows that weights of the form $\pm(\epsilon_j + \epsilon_{j'})$ does not belong to $\Psi(\mathbb{V}^{a_\sigma})$. We examine the assertion $\pm\epsilon_j, \epsilon_j - \epsilon_{j'} \in \Psi(\mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+)$ by the case-by-case analysis. We have three cases:

(Case $j, j' \in J_k$) We have $\epsilon_j - \epsilon'_j \in \Psi(\mathbb{V}^{a_\sigma})$ if and only if

$$\langle \epsilon_j - \epsilon_{j'}, s_\sigma \rangle = e^{(j'-j)\gamma} = e^\gamma.$$

This forces $j' - j = 1$. Hence, we put $i := j, j' = i+1$. We have $\langle \epsilon_i - \epsilon_{i+1}, D_\sigma \rangle = 1 \leq 1$, which verifies the first part of the assertion.

(Case $i \in J_k \neq J_m \ni j$) By (5.1), we deduce that

$$\langle \epsilon_j - \epsilon_{j'}, s_\sigma \rangle \in e^\Gamma = q_2^{\mathbb{Z}}$$

if and only if $j, j' \in J_k$ or $\delta_1(J_k) = \{0, 1\} = \delta_1(J_m)$ holds. We choose $j_0 \in J_k$ and $j_1 \in J_m$ such that $\delta_1(j_0) = 1 = \delta_1(j_1)$. We write $j = i + j_0$ and $j' := i' + j_1$. Then, we need

$$\langle \epsilon_{i+j_0} - \epsilon_{i'+j_1}, s_\sigma \rangle = e^{(i'-i)\gamma} = e^\gamma.$$

This happens if and only if $i' = i + 1$. By the definition of D_σ, we have

$$\langle \epsilon_{i+j_0} - \epsilon_{i+j_1+1}, D_\sigma \rangle \leq 1$$

if and only if $\underline{\#} J_k \geq \underline{\#} J_{k'}$. Since we assume $(\mathbf{J}, \vec{\delta})$ to be a strict marked partition, it follows that $\underline{\#} J_k \neq \underline{\#} J_{k'}$ by 1.13 4). This verifies the second part of the assertion.

(Case $j \in J_k$) If $\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$ or $-\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$, then we have $\langle s, \epsilon_j \rangle = e^{\gamma_0}$ or $e^{-\gamma_0}$, respectively. By (5.1), this forces $\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$ and $\delta_1(J_k) = \{0, 1\}$. Let $j_0 \in J_k$ be such that $\delta_1(j_0) = 1$. Put $j = i + j_0$ for some $i \in \mathbb{Z}$. Then, we have $\langle \epsilon_{i+j_0}, s_\sigma \rangle = e^{i\gamma+\gamma_0} = e^{\gamma_0}$ if and only if $i = 0$. Moreover, we have $\langle \epsilon_{j_0}, D_\sigma \rangle = 1 \leq 1$ and $j_0 > 0$, which verifies the final part of the assertion. $\square$

Lemma 5.6. *The group P_σ satisfies the following conditions:*

1. *$P_\sigma = TU_\sigma \subset B$, where U_σ is an unipotent subgroup of G;*
2. *The Lie algebra $\mathfrak{u}_\sigma$ of U_σ contains $\mathfrak{g}[\alpha] \subset \mathfrak{g}$ if and only if $\alpha = \epsilon_j - \epsilon_{j'}$, where j, j' are as follows:*
    - *$j \in J_k \neq J_{k'} \ni j'$ for some k, k';*
    - *There exists $j_0 \in J_k$ and $j_1 \in J_{k'}$ such that $\delta_1(j_0) = 1 = \delta_1(j_1)$;*
    - *$j - j_0 = j' - j_1$ and $j < j'$.*

Proof. Let L_σ be the reductive Levi component of P_σ which contains T. Let A be the Zariski closure of the group generated by D_σ and s_σ. Then, $G(A)$ is connected by Theorem 3.2. Thus, we have $L_\sigma = T$ if we have $\alpha(A) \neq \{1\}$ for each $\alpha \in R$. This is equivalent to $\alpha(D_\sigma) \neq 1$ or $\alpha(s_\sigma) \neq 1$ holds for each $\alpha \in R$ since R is a finite set. By (5.1), we have $\langle \epsilon_i - \epsilon_j, s_\sigma \rangle = 1$ only if

$$\exists \kappa \in \{\pm 1\} \text{ s.t. } \kappa i \in J_k \neq J_{k'} \ni \kappa j \text{ and } \delta_1(J_k) = \{0, 1\} = \delta_1(J_{k'}), \qquad (5.2)$$

where $J_k, J_{k'} \in \mathbf{J}$. By 1.13 4), we deduce that

$$\langle \epsilon_i, D_\sigma \rangle \neq \langle \epsilon_j, D_\sigma \rangle \text{ for each } i \in J_k, j \in J_{k'}.$$

Therefore, we conclude $L_\sigma = T$.

Since T normalizes the unipotent part of P_σ, we describe all one-parameter unipotent subgroup of G belonging to P_σ in order to prove the assertion. This is equivalent to count the set of weight spaces $\mathfrak{g}[\alpha] \subset \mathfrak{g}$ which is fixed by s_σ and has eigenvalue ≤ 1 with respect to D_σ. We examine the case $\alpha = \epsilon_i - \epsilon_j$ with the assumption (5.2) for $\kappa = +1$. (This last part of the assumption is achieved by swapping the roles of i and j if necessary.) Fix $j_0 \in J_k$ and $j_1 \in J_{k'}$ such that $\delta_1(j_0) = 1 = \delta_1(j_1)$. Then, the definition of s_σ further asserts $i - j_0 = j - j_1$. In order that D_σ has eigenvalue ≤ 1, we need to have

$$\langle \epsilon_i, D_\sigma \rangle \leq \langle \epsilon_j, D_\sigma \rangle ,$$

which is equivalent to $\underline{\#} J_k \geq \underline{\#} J_{k'}$. This implies $\# J_k > \# J_{k'}$ by 1.13 4). It follows that $i < j$, which verifies the second condition. Since $\alpha = \epsilon_i - \epsilon_j \in R^+$ in this case, we also deduce the first condition. $\square$

Proof of Proposition 5.4. Since $P_\sigma \subset G(s_\sigma)$, we have $P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{a_\sigma}$. Since the reductive part of P_σ is equal to T, we deduce $P_\sigma \mathbf{v}_\sigma \subset {}^{w_\sigma}\mathbb{V}^+$. Therefore, it suffices to prove the following equality at the level of tangent space

$$T_{\mathbf{v}_\sigma}(P_\sigma \mathbf{v}_\sigma) \cong \mathfrak{p}_\sigma \mathbf{v}_\sigma = \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+ \tag{5.3}$$

in order to deduce the assertion. Consider a T-weight decomposition $\mathbf{v}_\sigma^J = \sum_{\beta \in \Xi_J} v_\beta$, where $J \in \mathbf{J}$ and $0 \neq v_\beta \in \mathbb{V}[\beta]$. Each Ξ_J consists of linearly independent weights of $X^*(T_J)$. Moreover, we have $X^*(T_J) \cap X^*(T_{J'}) = \{0\}$ in $X^*(T)$ (by using the natural embeddings). Hence, we deduce

$$\mathfrak{t}\mathbf{v}_\sigma = \sum_{k \geq 1} \mathfrak{t}\mathbf{v}_\sigma^{J_k} = \sum_{k \geq 1} \sum_{\beta \in \Xi_{J_k}} \mathbb{C} v_\beta.$$

It is easy to see that $\bigcup_{k \geq 1} \Xi_{J_k}$ is precisely the set of T-weights described in Lemma 5.5 1) and 3).

In the below, we apply the action of $\mathfrak{u}_\sigma$ (c.f. Lemma 5.6) to fill out each $\mathbb{V}[\beta]$ for each T-weight β described in Lemma 5.5 2). Such a β is written as $\epsilon_i - \epsilon_j$, where $i \in J_k, j \in J_{k'}$ are as in Lemma 5.5 2). By explicit calculation, we have a non-zero element of $\mathfrak{g}$ of weight $\epsilon_{m+j_0} - \epsilon_{m+j_1}$ which satisfies

$$\xi_m \mathbf{v}_\sigma = \begin{cases} y_{m-1+j_0, m+j_1} - y_{m+j_0, m+j_1+1} & (m + j_0 + 1 \in J_{k'}) \\ y_{m-1+j_0, m+j_1} & (m + j_0 + 1 \notin J_{k'}) \end{cases}.$$

for each $m + j_0 \in J_{k'}$. (Here we implicitly used $\overline{\#} J_k > \overline{\#} J_{k'}$, which is deduced from $\underline{\#} J_k > \underline{\#} J_{k'}$ by 1.13 4).) We know $\xi_m \in \mathfrak{p}_\sigma$ by Lemma 5.6 2). We have

$$\left(\sum_{m \in \mathbb{Z}; m + j_1 \in J_{k'}} \mathbb{C} \xi_m \right) \mathbf{v}_\sigma = \sum_{m \in \mathbb{Z}; m + j_1 \in J_{k'}} \mathbb{V}[\epsilon_{m-1+j_0} - \epsilon_{m+j_1}].$$

By summing up for all possible pairs $(J_k, J_{k'}) \in \mathbf{J}$, the set of T-weights appearing in the RHS exhausts the T-weights described in Lemma 5.5 2). $\square$

Corollary 5.7. *Keep the setting of Proposition 5.4. Let $\nu = (a, \mathbf{v}_\sigma) = (s, \vec{q}, \mathbf{v}_\sigma)$ be an admissible parameter. Then, the natural embedding*

$$P_\sigma(s)\mathbf{v}_\sigma \subset \mathbb{V}^a \cap \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+$$

is dense open.

Proof. The assertion follows merely by taking a-fixed part of (5.3) in the proof of Proposition 5.4. $\square$

6 A vanishing theorem

We retain the setting of the previous section.

Definition 6.1 (Exotic Springer fibers)**.** For each $X \in \mathfrak{N}_2$, we define $\mathcal{E}_X$ as the image of the projection of

$$\boldsymbol{\mu}^{-1}(X) \subset \boldsymbol{F} = G \times^B \mathbb{V}^+$$

to G/B. For a pre-admissible parameter $\nu = (a, X)$, we have a subvariety $\boldsymbol{\mu}^{-1}(X)^a \subset \boldsymbol{\mu}^{-1}(X)$. We denote the image of $\boldsymbol{\mu}^{-1}(X)^a$ under the projection to G/B by $\mathcal{E}_X^a$. By construction, we have $\mathcal{E}_X \cong \boldsymbol{\mu}^{-1}(X)$ and $\mathcal{E}_X^a \cong \boldsymbol{\mu}^{-1}(X)^a \subset \boldsymbol{F}^a$. We call $\mathcal{E}_X$ and $\mathcal{E}_X^a$ exotic Springer fibers.

Theorem 6.2 (Homology vanishing theorem)**.** *Let $\nu = (a, X)$ be a pre-admissible parameter which is admissible or $a = a_0$. Then, we have*

$$H_{2i+1}(\mathcal{E}_X^a) = 0 \textit{ for every } i = 0, 1, \ldots.$$

Moreover, we have an isomorphism

$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_X^a) \xrightarrow{\cong} H_\bullet(\mathcal{E}_X^a).$$

Remark 6.3. **1)** The map ch in Theorem 6.2 is the homology Chern character map. (See e.g. [CG97] §5.8.) It sends the class of the (embedded) structure sheaf $\mathcal{O}_C$ for a closed subvariety $C \subset \mathcal{E}_X^a$ to

$$\mathsf{ch}[\mathcal{O}_C] = [C] + \text{lower degree terms} \in H_{2\dim C}(\mathcal{E}_X^a) \oplus \cdots \oplus H_0(\mathcal{E}_X^a).$$

2) The first part of Theorem 6.2 is valid even for integral coefficient case when $G(s) \subset GL(n, \mathbb{C})$ (c.f. [AH08])[2]. Here we present a proof along the line of earlier versions of this paper, with an enhancement (the proof of Theorem 6.2 modulo Proposition 6.7 given in §6.2) informed to the author by Eric Vasserot.

6.1 Review of general theory on homology vanishing

In this subsection, we recall several definitions and results of [BH85] and [DLP88] which we need in the course of our proof of Theorem 6.2.

Definition 6.4 (α-partitions)**.** A partition of a variety $\mathcal{X}$ over $\mathbb{C}$ is said to be an α-partition if it is indexed as $\mathcal{X}_1, \mathcal{X}_2, \ldots \mathcal{X}_k$ in such a way that $\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_i$ is closed for every $i = 1, \ldots, k$.

Theorem 6.5 ([DLP88] 1.7–1.10)**.** *Let $\mathcal{X}$ be a variety with α-partition $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$. If we have*

$$H_{2i+1}(\mathcal{X}_m) = 0 \textit{ for every } i = 0, 1, \ldots$$

[2] Previous versions of this paper also contain a similar result (since math.RT/0601155v3, April/2006). The author decided to drop it since it is unnecessary to prove our main theorems and [AH08] contains a better proof.

for each $m = 1, \ldots, k$, then we have

$$H_{2i+1}(\mathcal{X}) = 0 \text{ for every } i = 0, 1, \ldots.$$

Moreover, we have

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{X}) = \sum_{m \geq 1} \sum_{i \geq 0} \dim H_{2i}(\mathcal{X}_m).$$

Theorem 6.6 ([BH85] 9.1)**.** *Let $\mathcal{Z}$ be a smooth variety with $\mathbb{G}_m$-action. Assume that for some $t \in \mathbb{G}_m$, we have*

- $\mathcal{Z}^{\mathbb{G}_m} = \mathcal{Z}^t$ *and* $\lim_{N \to \infty} t^N z \in \mathcal{Z}^t \quad \forall z \in \mathcal{Z}$*;*
- *For each $z_0 \in \mathcal{Z}^t$, the set $\{z \in \mathcal{Z}; \lim_{N \to \infty} t^N z = z_0\}$ defines an affine closed subscheme of $\mathcal{Z}$.*

Then, $\mathcal{Z}$ is a vector bundle over $\mathcal{Z}^t$. In particular, the two conditions

$$H_{2i+1}(\mathcal{Z}) = 0 \text{ for every } i = 0, 1, \ldots, \text{ and } H_{2i+1}(\mathcal{Z}^t) = 0 \text{ for every } i = 0, 1, \ldots$$

are equivalent. Moreover, we have

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{Z}) = \sum_{i \geq 0} \dim H_{2i}(\mathcal{Z}^t)$$

if one of the above equivalent conditions hold.

6.2 Proof of vanishing theorem

This subsection is devoted to the proof of Theorem 6.2.

By taking G-conjugation if necessary, we assume $a \in \boldsymbol{T}$. We have

$$(a, X) = (s, \vec{q}, X) = \prod_{\mathbf{c} \in \mathcal{C}_a} (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}}).$$

By the same argument as in the proof of Corollary 3.10, each connected component of $\boldsymbol{\mu}^{-1}(X)^a$ is a product of connected components of

$$\mathcal{E}_{X_{\mathbf{c}}}^{(s_{\mathbf{c}}, \vec{q})} \subset Sp(2n^{\mathbf{c}})/(B \cap Sp(2n^{\mathbf{c}})) \text{ for all } \mathbf{c} \in \mathcal{C}_a.$$

Therefore, by the Künneth formula, it suffices to prove the assertion when $\mathcal{C}_a$ consists of a unique clan $[1, n]$. By Proposition 4.8, we further assume that $s \in T$, and $X = \mathbf{v}_\sigma$ for a strict marked partition $\sigma = (\mathbf{J}, \vec{\delta})$ by taking G-conjugate if necessary.

Proposition 6.7 (Weak version of Theorem 6.2)**.** *Let $\nu = (a, X) = (s, \vec{q}, \mathbf{v}_\sigma)$ be a pre-admissible parameter which is admissible or $a = a_0$. Assume that $s \in T$ and σ is a strict marked partition. For $s_\sigma \in T$ defined in the above of Lemma 5.2:*

- *We have*
$$H_{2i+1}((\mathcal{E}_X^a)^{s_\sigma}) = 0 \text{ for every } i = 0, 1, \ldots;$$

- *Each connected component of $(\mathcal{E}_X^a)^{s_\sigma}$ is smooth projective;*
- *We have an isomorphism*

$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K((\mathcal{E}_X^a)^{s_\sigma}) \xrightarrow{\cong} H_\bullet((\mathcal{E}_X^a)^{s_\sigma}).$$

Before giving the proof of Proposition 6.7, we complete our proof of Theorem 6.2 for $(a, \mathbf{v}_\sigma)$ assuming Proposition 6.7 for $(a, \mathbf{v}_\sigma)$.

Proof of Theorem 6.2 for $(a, X) = (a, \mathbf{v}_\sigma)$. Let $\mathcal{E}_1, \mathcal{E}_2, \ldots$ be a sequence of all connected components of $(\mathcal{E}_X^a)^{s_\sigma}$. For each $\mathcal{E}_k$, we set

$$\mathcal{B}_k := \{gB \in \mathcal{E}_X^a ; \lim_{N \to \infty} s_\sigma^{-N} gB \in \mathcal{E}_k\}.$$

Let $\pi_k : \mathcal{B}_k \to \mathcal{E}_k$ be the s_σ^{-1}-attracting map. Let

$$P := \{g \in G ; \lim_{N \to \infty} \mathrm{Ad}(s_\sigma^{-N}) g \in G\}$$

be a parabolic subgroup of G. It is straight-forward to see

$$\lim_{N \to \infty} \mathrm{Ad}(s_\sigma^{-N}) g \in G(s_\sigma)$$

for $g \in P$. It follows that each $\mathcal{B}_k$ intersects with a unique P-orbit in G/B. In particular, we can assume that the sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ forms an α-partition of $\bigcup_{k \ge 1} \mathcal{B}_k \subset \mathcal{E}_X^a$ by rearranging the sequence if necessary.

We choose $\{\gamma_i\}_i$ (in the definition of s_σ) so that we have

$$\begin{aligned} &\min\{\langle s_\sigma, \epsilon_i \rangle ; i \in J_k\} < \min\{\langle s_\sigma, \epsilon_i \rangle ; i \in J_{k'}\} \text{ , and} \\ &\gamma \max\{\langle s_\sigma, \epsilon_i \rangle ; i \in J_k\} > \max\{\langle s_\sigma, \epsilon_i \rangle ; i \in J_{k'}\} \end{aligned} \tag{6.1}$$

for each $k < k'$. (This choice is possible by $\# J_k \ge \# J_{k'}$ and Definition 1.13.)

Claim A. *Each fiber of π_k is an irreducible affine scheme.*

Proof. Let $P = G(s_\sigma) U$ be the Levi decomposition. Let $gB \in \mathcal{E}_k$. We set $F(gB) := \{u \in U(s) ; X \in ug\mathbb{V}^+\}$. This is a closed subset of $U(s)$. Set $U^\flat := (U(s) \cap gUg^{-1})$. We have a free right $U^\flat$-action on $F(gB)$. We have $\pi_k^{-1}(gB) \cong F(gB)/U^\flat$. Since $\pi_k^{-1}(gB)$ is a closed subspace of an affine space $U(s)/U^\flat$, it suffices to prove that $F(gB)$ is an affine space. We have a product decomposition $U(s) = U_2 U_1$, where U_1 is the product of $U_{\epsilon_i - \epsilon_j} \subset U(s)$ $(i, j > 0)$ and U_2 is the product of $U_{\epsilon_i + \epsilon_j} \subset U(s)$ $(i, j > 0)$. By (6.1), the space $(U_1 X - X)$ is a linear subspace of $\mathbb{V}^+$. Hence, $((U_1 X - X) \cap g\mathbb{V}^+)$ is an affine space. Here $\mathsf{Stab}_U X$ is a unipotent group, which is automatically an affine space.

Since U_2 acts V_1^+ trivially, we have $u \in F(gB)$ only if $u \in U_2 u_1$ with $u_1 \in F(gB)$. The closed subset $(U_2 u_1 \cap F(gB)) \subset U_2 u_1$ define linear conditions on U_2 since U_2 is commutative. Let $A \subset T$ denote the Zariski closure of the group generated by s and s_σ. The group $G(A)$ normalizes U_1 and U_2, and U_1 normalizes U_2. Hence, the condition along different points of $(U_1 \mathcal{E}_k \cap \mathcal{B}_k)$ are isomorphic via conjugation of $U_1 G(A)$. It follows that $F(gB)$ is a successive fibration of affine spaces by affine spaces, which is itself an affine space. $\square$

Claim B. *The variety $\mathcal{B}_k$ is a smooth affine bundle over $\mathcal{E}_k$.*

Proof. We keep the setting of the proof of Claim A. Let $\mathfrak{f}(gB) := \{\xi \in \mathrm{Lie}U(a); \xi X \in g\mathbb{V}^+\}$. Since gB is the unique s_σ-fixed point of $F(gB)$, we deduce

$$\dim F(gB) \leq \dim \mathfrak{f}(gB). \tag{6.2}$$

Notice that U is invariant under the $G(s_\sigma)$-action. Thus, $\dim \mathfrak{f}(gB)$ is invariant along $\mathcal{E}_k$. In view of Claim A, the assertion follows if the equality of (6.2) holds for each $gB \in \mathcal{E}_k$. If $\xi \in \mathfrak{f}(gB)$ is a s_σ-eigenvector, then we have $\exp(-\xi) = 1 - \xi \in F(gB)$. In particular, we have $\dim F(gB) \geq \dim \mathfrak{f}(gB)$ since we have enough number of linearly independent tangent lines. □

We return to the proof of Theorem 6.2

By Claim B and Theorem 6.6, the first assertion reduces to

$$H_{2i+1}(\mathcal{E}_k) = 0 \text{ for every } i = 0, 1, \ldots$$

for each k. Hence, Proposition 6.7 for (a, X) yields the first assertion.

Similarly, Proposition 6.7 for (a, X) and the Thom isomorphism give

$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{B}_k) \xrightarrow{\cong} H_\bullet(\mathcal{B}_k)$$

for each k. The Chern character map commutes with localization sequences. (c.f. [CG97] §5.8.) Therefore, a successive application of localization sequences implies the second assertion. □

The rest of this section is devoted to the proof of Proposition 6.7.

We set $(s_\sigma, \vec{q}_\sigma) := a_\sigma \in \boldsymbol{T}$ defined in §5. We have $a_\sigma X = X$. Hence, a_σ acts on $\mu^{-1}(X)^a$. Its projection gives the s_σ-action on $\mathcal{E}_X^a$. Let A be the Zariski closure of the subgroup of $\boldsymbol{T}$ generated by a and a_σ. We put $W_A := \{w \in W; B(A) \subset {}^wB\}$. We put $F^A(w) := G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^w\mathbb{V}^+)$ for each $w \in W_A$. We have $\bigcup_{w \in W_A} F^A(w) = (G \times^B \mathbb{V}^+)^A$. Consider the map

$${}^w\mu^A : F^A(w) = G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^w\mathbb{V}^+) \longrightarrow \mathbb{V}^A,$$

for each $w \in W_A$.

Lemma 6.8 (Part of Proposition 6.7)**.** *Each connected component of $(\mathcal{E}_X^a)^{a_\sigma}$ is smooth projective.*

Proof. Projectivity follows from that of $\mathcal{E}_X$, which itself follows by Theorem 1.2 3). By Lemma 5.6 1) and Corollary 5.7, we deduce that

$$\overline{B(A)\mathbf{v}_\sigma} \subset \mathbb{V}^A$$

is a linear subspace. It follows that $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ is a smooth subvariety of $G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^w\mathbb{V}^+)$. Hence, $({}^w\mu^A)^{-1}(B(A)\mathbf{v}_\sigma)$ is a smooth subvariety of $F^A(w)$. Since changing $\mathbf{v}_\sigma$ by $B(A)$-action gives isomorphic fibers, we deduce that $({}^w\mu^A)^{-1}(\mathbf{v}_\sigma)$ is a smooth subvariety of $F^A(w)$ as required. □

Corollary 6.9 (of the Proof of Lamma 6.8)**.** *The variety $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ is smooth.* □

We return to the proof of Proposition 6.7.

We prove the rest of assertions by the induction on the cardinality $n(\sigma)$ of the set

$$\mathtt{N}(\sigma) := \{J \in \mathbf{J}; \delta_1(J) = \{0,1\}\}.$$

In other words, we assume Theorem 6.2 for every admissible parameter of the form $(a, \mathbf{v}_{\sigma'})$ such that $n(\sigma') < n(\sigma)$. If $n(\sigma) = 0$, then Lemma 5.6 2) asserts that $G(s_\sigma) = T$. This implies that $(\mathcal{E}^a_X)^{s_\sigma}$ is a union of points. Thus, we obtain the assertion for $n(\sigma) = 0$.

We prove the assertion for $n(\sigma) = k$ by assuming that the assertion holds for all $n(\sigma) < k$. Let $J \in \mathtt{N}(\sigma)$ be the member such that $\# J \geq \# J'$ for every $J' \in \mathtt{N}(\sigma)$. Let σ' be a strict marked partition obtained from σ by replacing δ_1 by δ'_1 defined as:

$$\delta'_1(J) = \{0\}, \text{ and } \delta'_1(j) = \delta_1(j) \text{ for all } j \in [1,n] \backslash J.$$

Let $j_0 \in J$ be the unique element such that $\delta_1(j_0) = 1$. By Lemma 1.18, there exists $t \in T_J$ such that

$$\lim_{N \to \infty} t^N \mathbf{v}_\sigma = \mathbf{v}_\sigma - \mathbf{v}^J_{1,\sigma} = \mathbf{v}_{\sigma'}.$$

By Lemma 5.6 2), every T-weight of P_σ containing $i \in J$ is of the form $\epsilon_i - \epsilon_j$ for some $j \in J'$. Moreover, we have $P_{\sigma'} \subset P_\sigma$. It follows that the action of $t \in T$ contracts P_σ to $P_{\sigma'}$. By Corollary 5.7, the t-action also contracts $\mathbb{V}^A \cap {}^{w_\sigma}\mathbb{V}^+$ to

$$\mathbb{V}^A \cap \mathbb{V}^{a_{\sigma'}} \cap {}^{w_{\sigma'}}\mathbb{V}^+ = \mathbb{V}^{A'} \cap {}^{w_{\sigma'}}\mathbb{V}^+,$$

where A' is the Zariski closure of $\langle a, a_{\sigma'} \rangle \subset \boldsymbol{T}$.

Therefore, the t-action contracts $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ to $({}^w\mu^{A'})^{-1}(\overline{B(A')\mathbf{v}_{\sigma'}})$. By taking the quotient of

$$\mathcal{S} := B(A)\mathbf{v}_\sigma \cup B(A)\mathbf{v}_{\sigma'}$$

by $\mathsf{Stab}_{B(A)}\mathbf{v}^J_{1,\sigma}$, we obtain an affine plane $\mathbb{A}^1$ with contracting t-action to the origin. Therefore, we obtain a smooth family of smooth projective varieties over $\mathbb{A}^1$ whose fiber over $0 \in \mathbb{A}^1$ is $\mathcal{E}^A_{\mathbf{v}_{\sigma'}}$ and whose general fiber $\mathcal{E}^A_{\mathbf{v}_\sigma}$ contracting to $\mathcal{E}^{A'}_{\mathbf{v}_{\sigma'}}$. Moving smooth projective varieties is the same as moving all cycles by rational equivalence. Therefore, it suffices to prove

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}^A_{\mathbf{v}_{\sigma'}}) \xrightarrow{\cong} H_\bullet(\mathcal{E}^A_{\mathbf{v}_{\sigma'}}). \tag{6.3}$$

Since $\mathcal{E}^A_{\mathbf{v}_{\sigma'}}$ is smooth, the Białynicki-Birula theorem asserts that $\mathcal{E}^A_{\mathbf{v}_{\sigma'}}$ is a union of vector bundles over connected components of $\mathcal{E}^{A'}_{\mathbf{v}_{\sigma'}}$. Since the Chern character map commutes with pullbacks and localization sequences, we deduce (6.3) from Theorem 6.6 and Proposition 6.7 for $(a, \mathbf{v}_{\sigma'})$. Therefore, we have Proposition 6.7 for every admissible parameter of the form $(a, \mathbf{v}_\sigma)$ with $n(\sigma) = k$. Hence, the induction proceeds and we have proved Proposition 6.7 (and hence Theorem 6.2).

7 Standard modules and an induction theorem

We retain the setting of the previous section.

Definition 7.1 (Standard modules)**.** Let $\nu = (a, X)$ be a pre-admissible parameter. We define

$$M_\nu := H_\bullet(\mathcal{E}_X^a) \text{ and } M^\nu := H^\bullet(\mathcal{E}_X^a).$$

By the Ginzburg theory [CG97] 8.6, each of M_ν or M^ν is a $\mathbb{H}$-module.

By the symmetry of the construction of varieties involved in $M_{(a,X)}$ and $\mathbb{H}_a$, we deduce $M_{(a,X)} \cong M_{(\mathrm{Ad}(g)a, gX)}$ as $\mathbb{H}_a = \mathbb{H}_{\mathrm{Ad}(g)a}$-modules for each $g \in \boldsymbol{G}$.

Let $s_Q \in T(\mathbb{R})$ be an element such that

$$0 < \langle \alpha, s_Q \rangle \le 1 \text{ for all } \alpha \in R^+. \tag{7.1}$$

Let $Q := G(s_Q)$ and $\boldsymbol{Q} := Q \times (\mathbb{C}^\times)^3$. These are subgroups of G and $\boldsymbol{G}$, respectively. We put $\mathbb{V}_Q := \mathbb{V}_2^{s_Q}$ and $\mathfrak{N}_Q = \mathfrak{N}_2^{s_Q} \subset \mathbb{V}_Q$. We put $F_Q := Q \times^{(Q \cap B)} (\mathbb{V}_Q \cap \mathbb{V}_2^+)$. We have a map

$$\mu_Q : F_Q = Q \times^{(Q \cap B)} (\mathbb{V}_Q \cap \mathbb{V}_2^+) \longrightarrow \mathfrak{N}_Q.$$

We define $Z_Q := F_Q \times_{\mathfrak{N}_Q} F_Q$.

The natural inclusion map

$$F_Q = Q \times^{(Q \cap B)} (\mathbb{V}_Q \cap \mathbb{V}_2^+) \hookrightarrow \bigcup_{w \in W} Q \times^{({}^w B)(s_Q)} (\mathbb{V}_Q \cap {}^w\mathbb{V}_2^+) = \boldsymbol{F}^{s_Q}$$

gives an identification of F_Q with a connected component of $\boldsymbol{F}^{s_Q}$. This equips an action of $\boldsymbol{Q}$ on $\mathfrak{N}_Q$, F_Q, and Z_Q by restricting the $\boldsymbol{G}$-actions on their ambient spaces.

We put

$$\mathbb{H}_Q := \mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{Q}}(Z_Q),$$

where the convolution algebra structure on $K^{\boldsymbol{Q}}(Z_Q)$ are equipped by the restrictions of the maps p_1 and p_2 from $\boldsymbol{Z} \to \boldsymbol{F}$ to $Z_Q \to F_Q$.

Lemma 7.2. *Keep the above setting. Form an increasing sequence of integers*

$$1 \le n_1 \le n_2 \le \cdots$$

by requiring that

$$\alpha_i(s_Q) < 1 \textit{ if and only if } i = n_k \textit{ for some } k.$$

Then, we have

1. *$\mathbb{H}_Q$ is a subalgebra of $\mathbb{H}$ generated by $\mathcal{A}[T]$ and the set*
$$\{T_i; i \neq n_k \textit{ for some } k\};$$
2. *For a pre-admissible parameter $\nu = (s, \vec{q}, X)$ such that $s \in T$ and $X \in \mathfrak{N}_Q$, the vector space*
$$M_\nu^Q := H_\bullet(\mu_Q^{-1}(X)^{(s,\vec{q})})$$
is a $\mathbb{H}_Q$-module.

Proof. By the condition (7.1), we have $\langle \alpha + \beta, s_Q \rangle = 1$ for $\alpha, \beta \in R^+$ if and only if $\langle \alpha, s_Q \rangle = 1$ and $\langle \beta, s_Q \rangle = 1$. This implies that Q is generated by T and the one-parameter unipotent subgroups corresponding to simple roots α_i (and $-\alpha_i$) such that $\alpha_i(s_Q) = 1$.

The variety F_Q decomposes into a product of vector bundles over the flag varieties of simple components of Q. By explicit computation, we deduce that the vector bundles we concern are either **a)** the cotangent bundle of the flag variety when the simple component is type A, or **b)** the variety $\boldsymbol{F}$ for a (possibly smaller) symplectic group which arose as a simple component of Q. Moreover, the map μ_Q is the product of the moment maps of the cotangent bundles of flag varieties of type A and our map $\boldsymbol{\mu}$ (for some symplectic group).

Hence, taking account into the argument in §2, both statements are straightforward modifications of [CG97] §7.6 and §8.6. Thus, we leave the details to the reader. □

Corollary 7.3. *Under the assumption of Lemma 7.2, QB is a parabolic subgroup of G.*

Proof. See the first paragraph of the proof of Lemma 7.2. □

Let $\mathbb{V}_U$ be the unique T-equivariant splitting of the map $\mathbb{V}_2^+ \longrightarrow \mathbb{V}_2^+/\mathbb{V}_Q^+$. Let U be the unipotent radical of QB. If $X \in \mathbb{V}$ satisfies $s_Q X = X$, then s_Q has eigenvalue < 1 on $\mathfrak{u}X$. Hence, we have necessarily $\mathfrak{u}X \subset \mathbb{V}_U$.

For an admissible parameter $(s, \vec{q}, X)$, we can regard $X = (X_0 + X_1 \oplus X_2) \in \mathbb{V}$ as an element $X_0 \oplus X_1 \oplus X_2 \in \mathbb{V}_2$ so that $sX_i = q_i X_i \in V_1$ for $i = 0, 1$.

Theorem 7.4 (Induction theorem)**.** *We put $P := QB$. Let $P = QU$ be its Levi decomposition. Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter regarded as an element of $\boldsymbol{G} \times \mathbb{V}_2$. Assume $s \in Q$ and $X \in \mathfrak{N}_Q$. If we have*

$$\mathbb{V}_U^a \subset \mathfrak{u}X, \tag{7.2}$$

then we have an isomorphism

$$\mathsf{Ind}_{\mathbb{H}_Q}^{\mathbb{H}} M_\nu^Q \cong M_\nu$$

as $\mathbb{H}$-modules, where M_ν^Q is as in Lemma 7.2.

The rest of this section is devoted to the proof of Theorem 7.4.

By taking Q-conjugation if necessary, we assume $X \in \mathbb{V}_2^+$.

Let $W_Q := N_Q(T)/T \subset W$. We define

$$W^Q := \{w \in W; \ell(w) \le \ell(vw) \text{ for all } v \in W_Q\}.$$

Let $w \in W^Q$. Let $\mathcal{O}_w$ be the P-orbit of G/B which contains $\dot{w}B$. By counting the weights, we have $(\mathbb{V}_2^+ \cap \mathbb{V}_Q) \subset (\mathbb{V}_2^+ \cap {}^w\mathbb{V}_2^+)$. It follows that $X \in (\mathbb{V}_2^+ \cap {}^w\mathbb{V}_2^+)$. Hence, the map

$$(\mathcal{E}_X \cap \mathcal{O}_1) = \mu_Q^{-1}(X) \ni gB \mapsto g\dot{w}B \in \mathcal{E}_X \cap \mathcal{O}_w$$

gives rise to an isomorphism $(\mathcal{E}_X \cap \mathcal{O}_1) \cong \mathcal{E}_X \cap \mathcal{O}_w^{s_Q}$. Let B^- be the opposite Borel subgroup of B with respect to T. We put $U_w := U \cap {}^wB^-$. Since s_Q attracts points of $\mathcal{O}_w$, we obtain a map

$$\psi_w : \mathcal{E}_X \cap \mathcal{O}_w \to (\mathcal{E}_X \cap \mathcal{O}_w)^{s_Q} \cong (\mathcal{E}_X \cap \mathcal{O}_1)$$

which sends a point p to $\lim_{N\to\infty} s_Q^N \mathsf{p}$. We have an expression of a point $gu\dot{w}B \in \mathcal{E}_X \cap \mathcal{O}_w$ as $g \in Q$, $gB \in \mu_Q^{-1}(X)$, and $u \in U_w$. Let w_Q be the longest element of W^Q.

Lemma 7.5. *The fiber of the map ψ_w at $gB \in QB$ is given as*

$$\psi_w^{-1}(gB) = \{u \in gU_w g^{-1}; uX - X \in g^w \mathbb{V}_2^+ \cap \mathbb{V}_U\}.$$

In particular, $\psi_{w_Q}^{-1}(gB)$ is isomorphic to $\mathsf{Stab}_U(X)$.

Proof. The variety $\mathcal{O}_w$ is a U_w-fibration over $\mathcal{O}_1$. The condition $X \in gu\dot{w}\mathbb{V}_2^+$ is equivalent to $(gu^{-1}g^{-1})X - X \in g\dot{w}\mathbb{V}_2^+$. Moreover, U is Q-stable and $(gu^{-1}g^{-1})X - X \in \mathbb{V}_U$, which implies the first result. Since $U_{w_Q} = U$ and $g^{w_Q}\mathbb{V}_2^+ \cap \mathbb{V}_U = \{0\}$, we conclude the second assertion. □

Lemma 7.6. *We have*

$$\dim H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_w) = \dim H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1).$$

Proof. By the proof of Theorem 6.2 and [DLP88] 3.9, we have an α-partition $\mathcal{X}_1, \mathcal{X}_2, \ldots$ of $\mathcal{E}_X^a \cap \mathcal{O}_1$ such that each $\mathcal{X}_m$ $(m = 1, 2, \ldots)$ is a smooth variety without odd-term homology. By condition (7.2), we see that

$$\dim(g\dot{w}\mathbb{V}^+ \cap (X + \mathbb{V}_U^a)) = \dim(\dot{w}\mathbb{V}^+ \cap (X + \mathbb{V}_U^a))$$

for all $g = hu \in QU(s)$ such that $hB \in \mathcal{E}_X^a$. We denote its (common) dimension by d. Here $(g\dot{w}\mathbb{V}^+ \cap (X + \mathbb{V}_U^a))$ is an affine space contained in GX. The fiber of the map

$$\varphi_w : (\mathcal{E}_X^a \cap \mathcal{O}_w) \to (\mathcal{E}_X^a \cap \mathcal{O}_w^{s_Q}) \cong (\mathcal{E}_X^a \cap \mathcal{O}_1)$$

is isomorphic to a fiber of the following map at X:

$$U(s) \times^{(U(s) \cap \mathrm{Ad}(g\dot{w})B)} (g\dot{w}\mathbb{V}^+ \cap (X + \mathbb{V}_U^a)) \to X + \mathbb{V}_U^a = U(s)X.$$

In particular, φ_w is a smooth affine fibration of relative dimension $\dim U_w(s) + d - \dim \mathbb{V}_U^a$. Therefore, Theorem 6.6 implies that $\varphi_w^{-1}(\mathcal{X}_m)$ is a vector bundle over $\mathcal{X}_m$ for each m. Hence, we deduce that

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{X}_m) = \sum_{i \geq 0} \dim H_{2i}(\varphi_w^{-1}(\mathcal{X}_m))$$

and

$$H_{2i+1}(\mathcal{X}_m) = H_{2i+1}(\varphi_w^{-1}(\mathcal{X}_m)) = 0 \text{ for } i = 1, 2, \ldots$$

for each m. Since $\varphi_w^{-1}(\mathcal{X}_1), \varphi_w^{-1}(\mathcal{X}_2), \ldots$ forms an α-partition of $\mathcal{E}_X^a \cap \mathcal{O}_w$, we obtain the result. □

We return to the proof of Theorem 7.4.

It is easy to see that

$$\mathcal{E}_X^a = \bigsqcup_{w \in W^Q} (\mathcal{E}_X^a \cap \mathcal{O}_w)$$

forms an α-partition. Together with Theorem 6.2 and Lemma 7.6, this implies

$$\dim M_\nu = (\# W^Q) \dim M_\nu^Q = (\# W / \# W_Q) \dim M_\nu^Q. \tag{7.3}$$

Moreover, the natural map

$$\imath : M_\nu^Q = H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \hookrightarrow H_\bullet(\mathcal{E}_X^a) = M_\nu$$

is injective. Since we have

$$p_1(Z_{\le s_i} \cap p_2^{-1}(\mathcal{O}_1)) \subset \mathcal{O}_1 \text{ if } i \neq n_k \text{ for some } k = 1, 2, \dots,$$

the map $\imath$ is an embedding of $\mathbb{H}_Q$-modules. (The sequence $\{n_k\}_k$ is borrowed from Lemma 7.2.) Hence, we have an induced map

$$\phi : \mathsf{Ind}_{\mathbb{H}_Q}^{\mathbb{H}} M_\nu^Q \longrightarrow M_\nu.$$

Thanks to (7.3), we have:

Lemma 7.7. *Theorem 7.4 follows if ϕ is surjective.* □

We return to the proof of Theorem 7.4.

For each $w \in W^Q$, we define

$$R_w := [\mathcal{O}_{Z_{\le w^{-1}}}] \in K^{\boldsymbol{G}}(\boldsymbol{Z}).$$

By the construction of §2, we have

$$R_w H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \subset H_\bullet(\mathcal{E}_X^a \cap \overline{\mathcal{O}_w}) \subset H_\bullet(\mathcal{E}_X^a).$$

Since W^Q has a partial order $\le_Q$ induced by the Bruhat order, we put

$$H_\bullet(\mathcal{E}_X^a)_{\le w} := \sum_{v \le_Q w; v \in W^Q} R_v H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1).$$

Consider the composition map

$$\tau_w : H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \xrightarrow{R_w \circ} H_\bullet(\mathcal{E}_X^a \cap \overline{\mathcal{O}_w}) \xrightarrow{\mathsf{res}} H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_w).$$

Lemma 7.8. *Theorem 7.4 follows if each τ_w is surjective.*

Proof. We have

$$\dim \mathrm{Im}\phi \ge \sum_{w \in W^Q} \dim \mathsf{gr} H_\bullet(\mathcal{E}_X^a)_{\le w} = \sum_{w \in W^Q} \dim M_\nu^Q = (\# W^Q) \dim M_\nu^Q,$$

where we used the assumption at the first inequality. Here gr stands for the graded quotient with respect to some completed order on W^Q which extends $\le_Q$. By (7.3), we conclude that ϕ must be surjective. □

We return to the proof of Theorem 7.4.

We have only to prove that each τ_w is surjective provided if (7.2) holds. We have an open enbedding

$$\left(p_2^{-1}(\mathcal{E}_X \cap \mathcal{O}_1) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})\right) \subset \left(p_2^{-1}(\mathcal{E}_X \cap \mathcal{O}_1) \cap Z_{\le w^{-1}}\right).$$

Lemma 7.9. *For each subset $\mathcal{E} \subset (\mathcal{E}_X \cap \mathcal{O}_1)$, we have $\left(p_2^{-1}(\mathcal{E}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})\right) \cong \psi_w^{-1}(\mathcal{E})$.*

Proof. By definition, the LHS is written as:

$$\{(g_1B, g_2B) \in \boldsymbol{\mu}^{-1}(X) \times (\mathcal{E}_X \cap \mathcal{O}_1); g_1^{-1}g_2 \in B\dot{w}^{-1}B\}.$$

Since $B \cap Q = {}^{w}B \cap Q$, we have $B\dot{w}^{-1}B = B\dot{w}^{-1}U$. By taking the right B-translation if necessary, we can assume $g_1 \in g_2U\dot{w}$. This forces g_1B to live in the fiber of the map ψ_w. This implies that g_2B is completely determined by the data of $\psi_w^{-1}(\mathcal{E})$ and vice versa. $\square$

Let A be the Zariski closure of $\langle a, s_Q\rangle \subset \boldsymbol{T}$. The set $\mathfrak{u}X \subset \mathbb{V}_U$ is an A-stable linear subspace. It follows that

$$S := \mathbb{V}_U/\mathfrak{u}X$$

has a A-stable splitting in $\mathbb{V}_U$. Using this splitting, we define

$$\mathcal{E}_X^{\sim} := \{(gB, X+y) \in \boldsymbol{F}; gB \in (\mathcal{E}_X \cap \mathcal{O}_1), y \in S\}.$$

Each element of $\mathbb{V}_U$ is contracted to 0 by the s_Q-action. Hence, S has a contraction to $0 \in S$. This gives a contraction

$$\theta : \mathcal{E}_X^{\sim} \longrightarrow (\mathcal{E}_X \cap \mathcal{O}_1)$$

given by collecting s_Q-attracting points.

Theorem 7.10. *For each $w \in W^Q$, the intersection of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$ and $(\boldsymbol{F} \times \mathcal{E}_X^{\sim})$ is transversal inside $\boldsymbol{F}^2$.*

Proof. We prove the assertion by induction. The case $w = 1$ is clear. Assume that

- $w = w's$ by $w' \in W^Q$ and $s = s_i$ such that $\ell(w) = \ell(w') + 1$;
- The assertion holds for w';

and prove the assertion for w. Let δ_{in} be Kronecker's delta which takes 1 if $s = s_n$ and 0 otherwise. For $v = w, w'$, we set

$$\mathcal{E}^{\sim}(v) := \{(g\dot{v}B, X+y) \in p_1((\boldsymbol{F} \times \mathcal{E}_X^{\sim}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{v^{-1}}))\}.$$

We denote the fibers of the maps $\mathcal{E}^{\sim}(v) \to (\mathcal{E}_X \cap \mathcal{O}_v^{s_Q})$ over $g\dot{v}B$ as:

$$F_v(gB) := \{(u, X+y) \in gU_vg^{-1} \times S; u^{-1}X - X \in g^v\mathbb{V}_2^+, y \in S \cap gu^v\mathbb{V}_2^+\}.$$

We have

$$\dim \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}}) = \dim \boldsymbol{\pi}^{-1}(\mathsf{O}_{(w')^{-1}}) - \delta_{in}.$$

Taking account into Lemma 7.9, the failure of the dimension condition of transversal intersection implies

$$\begin{aligned} &\dim F_w(gB) > \dim F_{w'}(gB) \text{ if } s \neq s_n \text{ or} \\ &\dim F_w(gB) \geq \dim F_{w'}(gB) \text{ if } s = s_n \end{aligned} \tag{7.4}$$

for some $gB \in (\mathcal{E}_X \cap \mathcal{O}_1)$. We assume (7.4) (for some gB) to deduce contradiction. Being transversal intersection is an open condition. Since the intersection

has a contraction to its s_Q-fixed point, it suffices to consider the situation near s_Q-fixed points. Hence, we replace $F_v(gB)$ in (7.4) by the following tangent space version

$$\mathfrak{f}_v(gB) := \{(\xi, y) \in \mathrm{Ad}(g)\mathfrak{u}_v \times S; \xi X \in g^v \mathbb{V}_2^+, y \in S \cap g^v \mathbb{V}_2^+\}.$$

Let $\mathfrak{u}^v := \mathrm{Lie}(U \cap {}^v B)$. It is clear that $(\mathrm{Ad}(g)\mathfrak{u}^v)X \subset g^v \mathbb{V}_2^+$ since $X \in g^v \mathbb{V}_2^+$. We have $\mathfrak{u} = \mathfrak{u}^v \oplus \mathfrak{u}_v$. We put $\Delta(v) := \dim(\mathfrak{u}^v Y \cap \mathfrak{u}_v Y)$. It follows that

$$\dim \mathfrak{f}_v(gB) = \dim \mathsf{Stab}_{\mathfrak{u}_v} Y + \dim(\mathbb{V}_U \cap {}^v \mathbb{V}_2^+)/\mathfrak{u}^v Y + \Delta(v)$$

where we put $Y := g^{-1}X$. We have

$$\mathfrak{u}_w = \mathfrak{u}_{w'} \oplus \mathfrak{g}[\alpha], \text{ and } \mathfrak{u}^w \oplus \mathfrak{g}[\alpha] = \mathfrak{u}^{w'}$$

for $\alpha = w'\alpha_i$. According to the behavior of $\mathfrak{g}[\alpha]Y$, we have four cases:

(Case $\mathfrak{g}[\alpha]Y \subset \mathfrak{u}^w Y \cap \mathfrak{u}_{w'} Y$) We have

$$\begin{aligned}1 + \delta_{in} + \dim(\mathbb{V}_U \cap {}^w \mathbb{V}_2^+)/\mathfrak{u}^w Y &= \dim(\mathbb{V}_U \cap {}^{w'} \mathbb{V}_2^+)/\mathfrak{u}^{w'} Y,\\ \dim \mathsf{Stab}_{\mathfrak{u}_w} Y - 1 &= \dim \mathsf{Stab}_{\mathfrak{u}_{w'}} Y, \text{ and } \Delta(w) = \Delta(w').\end{aligned}$$

(Case $\mathfrak{u}^w Y \supset \mathfrak{g}[\alpha]Y \not\subset \mathfrak{u}_{w'} Y$) We have

$$\begin{aligned}1 + \delta_{in} + \dim(\mathbb{V}_U \cap {}^w \mathbb{V}_2^+)/\mathfrak{u}^w Y &= \dim(\mathbb{V}_U \cap {}^{w'} \mathbb{V}_2^+)/\mathfrak{u}^{w'} Y,\\ \dim \mathsf{Stab}_{\mathfrak{u}_w} Y &= \dim \mathsf{Stab}_{\mathfrak{u}_{w'}} Y, \text{ and } \Delta(w) - 1 = \Delta(w').\end{aligned}$$

(Case $\mathfrak{u}^w Y \not\supset \mathfrak{g}[\alpha]Y \subset \mathfrak{u}_{w'} Y$)

$$\begin{aligned}\delta_{in} + \dim(\mathbb{V}_U \cap {}^w \mathbb{V}_2^+)/\mathfrak{u}^w Y &= \dim(\mathbb{V}_U \cap {}^{w'} \mathbb{V}_2^+)/\mathfrak{u}^{w'} Y,\\ \dim \mathsf{Stab}_{\mathfrak{u}_w} Y - 1 &= \dim \mathsf{Stab}_{\mathfrak{u}_{w'}} Y, \text{ and } \Delta(w) + 1 = \Delta(w').\end{aligned}$$

(Case $\mathfrak{u}^w Y \not\supset \mathfrak{g}[\alpha]Y \not\subset \mathfrak{u}_{w'} Y$)

$$\begin{aligned}\delta_{in} + \dim(\mathbb{V}_U \cap {}^w \mathbb{V}_2^+)/\mathfrak{u}^w Y &= \dim(\mathbb{V}_U \cap {}^{w'} \mathbb{V}_2^+)/\mathfrak{u}^{w'} Y,\\ \dim \mathsf{Stab}_{\mathfrak{u}_w} Y &= \dim \mathsf{Stab}_{\mathfrak{u}_{w'}} Y, \text{ and } \Delta(w) = \Delta(w').\end{aligned}$$

This case-by-case analysis claims that we cannot achieve the infinitesimal version of (7.4). Hence, we have contradiction. It follows that the intersection of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$ and $(\boldsymbol{F} \times \mathcal{E}_X^{\sim})$ must have proper dimension inside $\boldsymbol{F}^2$ under the induction hypothesis.

Now the linear independence of the normal vectors follows as an immediate consequence of the fact that they are concentrated on the first factor and S on the second factor (of $\boldsymbol{F} \times \mathcal{E}_X^{\sim}$), or the diagonal part (of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$), respectively.

Therefore, the induction proceeds and we obtain the result. $\square$

Lemma 7.11. *The map τ_w is an isomorphism.*

Proof. By [CG97] 2.7.26 and Theorem 7.10, we deduce that the map τ_w induces an isomorphism

$$H_\bullet(\mathcal{E}_X^{\sim}) \xrightarrow{\cong} H_\bullet(p_1((\boldsymbol{F} \times \mathcal{E}_X^{\sim}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}}))). \tag{7.5}$$

The spaces appearing in the homologies are given as fibrations over $(\mathcal{E}_X \cap \mathcal{O}_1)$ and $(\mathcal{E}_X \cap \mathcal{O}_w)$ with its fiber linear subspaces of S. Here $\mathcal{E}_X^{\sim}$ has larger fiber. We switch to algebraic K-theory by Theorem 6.2 and Lemma 7.9. We have

$$R(A)_a \otimes_{R(A)} K^A(\mathcal{E}_X^{\sim}) \cong R(A)_a \otimes_{R(A)} K^A(\mathcal{E}_X^a \cap \mathcal{O}_1) = R(A)_a \otimes_{\mathbb{Z}} K(\mathcal{E}_X^a \cap \mathcal{O}_1)$$

by the Thomason localization theorem and the fact that A fixes $(\mathcal{E}_X^a \cap \mathcal{O}_1)$. Hence, the map τ_w itself is surjective if $[Y], [\theta^{-1}(Y)] \in R(A)_a \otimes_{R(A)} K^A(\mathcal{E}_X^{\sim})$ define the same cycle up to an invertible factor for each A-stable closed subvariety $Y \subset (\mathcal{E}_X \cap \mathcal{O}_1)$. This is true if the alternating sum of the Koszul complex of S is invertible in $R(A)_a$. This is equivalent to $S^a = 0$, which is further re-phrased as

$$\mathbb{V}_U^a \subset \mathfrak{u}X.$$

This is (7.2). $\square$

We return to the proof of Theorem 7.4.

Thanks to Lemma 7.11, we have finished the proof of Theorem 7.4 by Lemma 7.8.

8 Exotic Springer correspondence

We keep the setting of the previous section.

As we see in §1.4, we know that the action of $\mathbb{H}$ on M_ν factors through the isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a) \xrightarrow{\mathrm{RR}} H_\bullet(\boldsymbol{Z}^a) \cong \mathbb{H}_a.$$

Let $\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle$ be the ideal of $\mathbb{C}[\mathfrak{t}]$ generated by the set of W-invariant polynomials without constant terms. Let $\mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle\right)$ be the smash-product, which means that its product is given as

$$(w_1, f_1)(w_2, f_2) := (w_1 w_2, f_1 w_1(f_2)) \text{ for } w_1, w_2 \in W, f_1, f_2 \in \mathbb{C}[\mathfrak{t}]/\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle.$$

It is clear that $\boldsymbol{F}^{a_0} \cong F$, $\boldsymbol{Z}^{a_0} \cong Z$, and the restriction of the natural projections $\boldsymbol{Z} \to \boldsymbol{F}$ restrict to natural projections $Z \to F$.

Proposition 8.1. *We have an isomorphism*

$$\mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle\right) \cong H_\bullet(\boldsymbol{Z}^{a_0})$$

as algebras.

Proof. We have

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C}_{a_0} \otimes_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(Z).$$

Here the RHS is written as

$$\mathbb{C} \otimes_{R(G)} \mathbb{H}/(\mathbf{q}_0 = -\mathbf{q}_1 = \mathbf{q}_2 = 1).$$

Thus, we have

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C} \otimes_{R(G)} \mathbb{C}[\widetilde{W}],$$

where $\widetilde{W} := W \ltimes X^*(T)$ is the affine Weyl group of type $C_n^{(1)}$. (Here $\mathbb{C}$ is the $R(G)$-module given by the evaluation at $1 \in G$. The algebra $R(G)$ acts on $\mathbb{C}[\widetilde{W}]$ by $R(G) \cong \mathbb{Z}[X^*(T)]^W$.) Thus, it suffices to show

$$\mathbb{C}[X^*(T)]/\mathbb{C}[X^*(T)]\mathfrak{m}_1^{\sim} \cong \mathbb{C}[\mathfrak{t}]/\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle,$$

where $\mathfrak{m}_1^{\sim} \subset \mathbb{C}[X^*(T)]^W = \mathbb{C}[T]^W$ is the defining ideal of the image of $1 \in T$ in $\mathrm{Spec}\mathbb{C}[T]^W$. This follows from the fact that the neighborhoods of $1 \in T$ and $0 \in \mathfrak{t}$ are W-equivariantly diffeomorphic through the exponential map. □

Corollary 8.2. *Keep the setting of 8.1. We have a surjection*

$$H_\bullet(\boldsymbol{Z}^{a_0}) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[W].$$

Proof. Keep the notation of the proof of Theorem 8.1. We have

$$\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle \subset \mathfrak{m}_1 \subset \mathbb{C}[\mathfrak{t}],$$

where $\mathfrak{m}_1$ is the defining ideal of $0 \in \mathfrak{t}$. Since 0 is a W-fixed point of $\mathfrak{t}$, we deduce that $\mathfrak{m}_1$ is a W-invariant maximal ideal. It follows that

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\left\langle \mathbb{C}[\mathfrak{t}]_+^W \right\rangle\right) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\mathfrak{m}_1\right) \cong \mathbb{C}[W]$$

as desired. □

Theorem 8.3 (Exotic Springer correspondence)**.** *There exist one-to-one correspondences between the sets of the following three kinds of objects:*

- *a strict marked partition σ;*
- *the G-orbit of $\mathfrak{N}$ given as $G\mathbf{v}_\sigma$;*
- *an irreducible W-module.*

Remark 8.4. Our proof of Theorem 8.3 does not tell which representation is obtained from a given orbit. Such information can be found in [Ka08], which employs totally different argument.

Proof of Theorem 8.3. Let $\mathcal{P}$ be the set of isomorphism classes of G-equivariant irreducible perverse sheaves on $\mathfrak{N}$. Each $I \in \mathcal{P}$ is isomorphic to the minimal extension from a G-orbit of $\mathfrak{N}$. By Proposition 4.5, the (perverse) sheaf I must be the extension of a constant sheaf on a G-orbit. This implies $\#\mathcal{P} \le \#(G\backslash\mathfrak{N})$. Let S be the set of strict normal forms. By Proposition 1.16 1), we have $\#(G\backslash\mathfrak{N}) \le \#S$. Hence, we have

$$\#\mathsf{Irrep}W \le \#\mathcal{P} \le \#(G\backslash\mathfrak{N}) \le \#S \le \#\mathsf{Irrep}W, \tag{8.1}$$

where the first inequality comes from Theorem 1.20 and the last inequality is Proposition 1.16 2). This forces all the inequalities in (8.1) to be equalities as required. □

The following is a summary of the consequences of §1.4 Theorem 1.20:

Theorem 8.5 (Ginzburg, [CG97] §8.5)**.** *Let a be a finite pre-admissible element. Let L be an irreducible $\mathbb{H}_a$-module. Then, there exists a unique $\boldsymbol{G}(a)$-orbit $\mathcal{O} \subset \mathfrak{N}^a$ with the following properties:*

1. *There exists a surjective $\mathbb{H}_a$-module homomorphism $M_{(a,X)} \to L$ for every $X \in \mathcal{O}$;*

2. *If L appears in the composition factor of $M_{(a,Y)}$ as $\mathbb{H}_a$-modules for some $Y \in \mathfrak{N}^a$, then we have $Y \in \overline{\mathcal{O}}$.* $\Box$

Theorems 8.3 and 8.5 claim that each strict marked partition σ gives a unique simple quotient of $M_{(a_0,\mathbf{v}_\sigma)}$. We denote this W-module by L_σ or L_X for $X \in G\mathbf{v}_\sigma$, depending on the situation.

Corollary 8.6. *Keep the setting of Theorem 8.3. A $\mathbb{C}[W]$-module $M_{(a_0,X)}$ contains L_σ only if $X \in \overline{G\mathbf{v}_\sigma}$ holds.* $\Box$

9 A deformation argument on parameters

We retain the setting of the previous section.

Theorem 9.1. *Let $a = (s, \vec{q}) \in \boldsymbol{T}$ be an admissible element such that $\mathcal{C}_a = \{[1,n]\}$. Then, there exists an admissible element $a' := (s', \vec{q}')$ such that*

- *The s'-action on V_1 has only positive real eigenvalues;*
- *We have $q'_0, q'_1, q'_2 \in \mathbb{R}^\times_{>0}$;*
- *We have equalities $\mathfrak{N}^a_+ = \mathfrak{N}^{a'}_+$ and $G(s) = G(s')$.*

Moreover, we have an isomorphism $\mathbb{H}_a \cong \mathbb{H}_{a'}$ as algebras.

Proof. Let N be the largest positive integer such that $1, q_2, \ldots, q_2^N$ are distinct. (If q_2 is not a root of unity, then we regard $N = \infty$.) For each $i = 1, \ldots, n$, we set $\chi_i := \epsilon_i(s)$. By rearranging s by the W-action if necessary, we assume $|\chi_i| \geq 1$ (if $N = \infty$) or $\chi_i = q_2^j$ for some $j \in \frac{1}{2}[0, N]$. We set $\mathtt{E} := \{\chi_i; 1 \leq i \leq n\}$. We choose a representative $j_0 \in [1, n]$ which satisfies the following condition:

- If $\pm 1 \in \mathtt{E}$, then we require $\chi_{j_0} = \pm 1$;
- If $\pm q_2^{1/2} \in \mathtt{E}$, then we require $\chi_{j_0} = \pm q_2^{1/2}$;
- If $\pm q_2^{1/2} \notin \mathtt{E}$ and $\pm q_2^{-1/2} \in \mathtt{E}$, then we require $\chi_{j_0} = \pm q_2^{-1/2}$;

For each pair $i, j \in [1, n]$, we have

$$\chi_i^{\kappa_{i,j}} = \chi_j^{\kappa'_{i,j}} q_2^{m_{i,j}} \text{ for some } \kappa_{i,j}, \kappa'_{i,j} \in \{\pm 1\}, m_{i,j} \in [0, n]. \tag{9.1}$$

Since q_2 is not a root of unity of order $\leq 2n$, it follows that the choice of $m_{i,j}$ is at most one if $(\kappa_{i,j}, \kappa'_{i,j})$ is fixed. For each pair (i, j) in $[1, n]$, we set $\mathtt{I}_{(i,j)}$ to be the set of triples $(\kappa_{i,j}, \kappa'_{i,j}, m_{i,j})$ which satisfies (9.1). Choose two real numbers $q \gg q'_2 \gg 1$ such that q and q'_2 have no algebraic relation. Then, we set

$$(\chi'_i)^{\kappa_{i,j_0}} := \begin{cases} (q'_2)^{m_{i,j_0}} & (\chi_{j_0} = \pm 1) \\ (q'_2)^{m_{i,j_0} + \kappa'_{i,j_0}/2} & (\chi_{j_0} = \pm q_2^{1/2}) \\ (q'_2)^{m_{i,j_0} - \kappa'_{i,j_0}/2} & (\chi_{j_0} = \pm q_2^{-1/2}) \\ q(q'_2)^{m_{i,j_0}} & (\chi_{j_0} \neq \pm 1, \pm q_2^{\pm 1/2}) \end{cases}.$$

Since the relation (9.1) for (i,j) is determined by that of (i,j_0) and (j,j_0) for each pair i,j in $[1,n]$, it follows that

$$(\chi_i')^{\kappa_{i,j}} = (\chi_j')^{\kappa_{i,j}'}(q_2')^{m_{i,j}} \text{ for some } \kappa_{i,j}, \kappa_{i,j}' \in \{\pm 1\}, m_{i,j} \in [0,n] \tag{9.2}$$

for all $(\kappa_{i,j}, \kappa_{i,j}', m_{i,j}) \in \mathtt{I}_{(i,j)}$. Conversely, we have $(\kappa_{i,j}, \kappa_{i,j}', m_{i,j}) \in \mathtt{I}_{(i,j)}$ if the relation (9.2) holds. It is clear that $\chi_i^2 = 1$ if and only if $(\chi_i')^2 = 1$. We put $s' \in T$ so that $\epsilon_i(s') = \chi_i'$ for each $i = 1, 2, \ldots, n$. By the above consideration, it follows that $\mathfrak{g}(s') = \mathfrak{g}(s)$. Since both $G(s')$ and $G(s)$ are connected by Theorem 3.2, we deduce $G(s) = G(s')$.

Since the relation of (9.1) is preserved, we have $V_2^{(s,q_2)} = V_2^{(s',q_2')}$. If we have $\chi_i^{\kappa_i} = q_k$ for some $i \in [1,n]$, $\kappa_i \in \{\pm 1\}$, and $k = 0, 1$, then we set $q_k' := (\chi_i')^{\kappa_i}$. Otherwise, we set q_k' $(i = 0, 1)$ to be an arbitrary real number which is not an eigenvalue of s' on V_1. (I.e. not equal to any of $(\chi_i')^{\pm 1}$.) Since we have infinitely many possibilities, we can assume $q_0' \neq q_1'$ and $q_k' \gg 1$ in this case. This gives $\mathbb{V}^a = \mathbb{V}^{a'}$ by setting $a' := (s', \vec{q}')$. We have $q_0' \neq q_1'$ in all cases since $q_0 \neq q_1$. Hence, the isomorphism $\mathbb{V}_2^a \cong \mathbb{V}^a$ implies $\mathbb{V}_2^{a'} \cong \mathbb{V}^{a'}$.

Therefore, as subvarieties of $\boldsymbol{F}$ and $\mathfrak{N}_2$, we have equalities

$$\boldsymbol{F}^a = \bigcup_{w \in W} G(s) \times^{{}^w B(s)} ({}^w\mathbb{V}^+ \cap \mathbb{V}^a) = \bigcup_{w \in W} G(s') \times^{{}^w B(s')} ({}^w\mathbb{V}^+ \cap \mathbb{V}^{a'}) = \boldsymbol{F}^{a'}$$

and $\mathfrak{N}_+^a = \mathfrak{N}_+^{a'}$.

The projection map $\boldsymbol{F}^a \to \mathfrak{N}_+^a$ is induced by the projection $\boldsymbol{\mu}$. Hence, so is $\boldsymbol{F}^{a'} \to \mathfrak{N}_+^{a'}$. Therefore, we have an equality of convolution algebras

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a) = \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^{a'}) \cong \mathbb{H}_{a'},$$

which proves the last assertion.

Since $q, q_2' \gg 1$, each of q_i' $(i = 0, 1)$ is positive real. This verifies the requirement about $\vec{q}'$ as desired. $\square$

Proposition 9.2. *Let $a = (s, q_0, q_1, q_2) \in \boldsymbol{T}$ be an admissible element such that:*

- *We have $\mathcal{C}_a = \{[1,n]\}$;*
- *The s-action on V_1 has only positive real eigenvalues;*
- *We have $V_1^{(s,q_1)} = \{0\}$;*
- *Each q_i $(i = 0, 1, 2)$ is a positive real number;*

Let $\underline{a} := (s, q_0, q_2)$ and let

$$\log \underline{a} := (\log s, r_0, r_2), \text{ where } q_0 = e^{r_0}, q_2 = e^{r_2}.$$

Let A be the Zariski closure of $\langle \underline{a} \rangle \subset \boldsymbol{T}$. Then $H_\bullet^A(Z)$ is a $\mathbb{C}[\mathfrak{a}]$-algebra such that

1. *The quotient of $H_\bullet^A(Z)$ by the ideal generated by functions of $\mathbb{C}[\mathfrak{a}]$ which is zero along $\log \underline{a}$ is isomorphic to $H_\bullet(Z^{\underline{a}})$;*

2. The images of the natural inclusions $\mathbb{C}[W] \subset H_\bullet(Z) \subset H^A_\bullet(Z)$ induces an injection

$$\mathbb{C}[W] \hookrightarrow H_\bullet(Z^{\underline{a}}) = H_\bullet(\boldsymbol{Z}^a).$$

Moreover, we have

$$\mathbb{C}[\mathfrak{a}] \otimes H_\bullet(\mathcal{E}_X) \cong H^A_\bullet(\mathcal{E}_X) \text{ for } X \in \mathfrak{N}^{\underline{a}}$$

as a compatible $(\mathbb{C}[W], \mathbb{C}[\mathfrak{a}])$-module, where W acts on $\mathfrak{a}$ trivially.

Corollary 9.3. *Keep the setting of Proposition 9.2. We have*

$$M_{(a_0,X)} = H_\bullet(\mathcal{E}_X) \cong H_\bullet(\mathcal{E}^a_X) = M_{(a,X)}$$

as $\mathbb{C}[W]$-modules. □

The rest of this section is devoted to the proof of Proposition 9.2.

Lemma 9.4. *Keep the setting of Proposition 9.2. Then, A is connected.*

Proof. The group A is defined to be the spectrum of the quotient of $\mathbb{C}[T_1]$ by the ideal generated by monomials m such that $m(s, q_0, q_2) = 1$. Since all the values of $\epsilon_i(s)$, q_0, q_1 are positive real number, the conditions $m(s, q_0, q_2) = 1$ and $m(s^r, q_0^r, q_2^r) = 1$ are the same for all $r \in \mathbb{R}_{>0}$, where the branch of powers are taken so that all of $\epsilon_i(s^r), q_0^r, q_2^r$ $(i = 1, \ldots, n)$ are positive real numbers. It follows that a monomial $m \in \mathbb{C}[T_1]$ satisfies $m(s, q_0, q_2)^k = 1$ for some positive integer k if and only if $m(s, q_0, q_2) = 1$. Therefore, such monomials form a saturated $\mathbb{Z}$-sublattice of $X^*(T_1)$. In particular, its quotient lattice is a free $\mathbb{Z}$-lattice, which implies that A is connected. □

We return to the proof of Proposition 9.2.

For each $m \geq 0$, let $ET_m := (\mathbb{C}^m \backslash \{0\})^{\dim \boldsymbol{T}}$ be a variety such that i-th $\mathbb{C}^\times$-factor of $\boldsymbol{T} = (\mathbb{C}^\times)^{\dim \boldsymbol{T}}$ acts as dilation of the i-th factor for each $1 \leq i \leq n+3$. By the standard embedding $\mathbb{C}^m \hookrightarrow \mathbb{C}^{m+1}$ sending (x) to $(x, 0)$, we form a sequence of $\boldsymbol{T}$-varieties

$$\emptyset = ET_0 \hookrightarrow ET_1 \hookrightarrow ET_2 \hookrightarrow \cdots. \tag{9.3}$$

We define $ET := \varinjlim_m ET_m$, which is an ind-quasiaffine scheme with free $\boldsymbol{T}$-action. When we consider the homology of ET, we refer to the homology of its underlying classical topological space $\bigcup_{m \geq 0} ET_m$. Since ET is contractible manifold with respect to the classical topology, we regard ET as the universal vector bundle of each subgroup of $\boldsymbol{T}$. (Hence we regard $BA := A \backslash ET$ in the below.)

Corollary 9.5 (of Lemma 9.4)**.** *Keep the above setting. We have $H^{odd}(BA) = 0$.* □

We return to the proof of Proposition 9.2.

For a A-variety $\mathcal{X}$, we set

$$\mathcal{X}_A := \triangle A \backslash (ET \times \mathcal{X}),$$

where $\triangle A$ represents the diagonal action of A. We have a forgetful map

$$f^A_{\mathcal{X}} : \mathcal{X}_A \to BA = A \backslash ET.$$

Let $\mathbb{D}^A_{\mathcal{X}}$ be the relative dualizing sheaf with respect to $f^A_{\mathcal{X}}$ (c.f. Bernstein-Lunts [BL94] §1.6). We define

$$H^A_i(\mathcal{X}) \cong H^{-i}(\mathcal{X}_A, \mathbb{D}^A_{\mathcal{X}}).$$

In the below, we understand that $H^A_\bullet(\mathcal{X}) := \bigoplus_m H^A_m(\mathcal{X})$. Notice that this homology group is the same as the one obtained by replacing ET with an ind-object of the direct system $\{ET_m\}$ and take the limit of the associated inverse system since $\mathcal{X}$ is homotopic to a finite dimensional CW-complex. The projection maps $p_i : Z_A \to F_A$ $(i = 1, 2)$ equip $H^A_\bullet(Z)$ a structure of convolution algebra. It is straight-forward to see that the diagonal subsets $\triangle F \subset Z$ and $(\triangle F)_A \subset Z_A$ represents $1 \in H_\bullet(Z)$ and $1 \in H^A_\bullet(Z)$, respectively.

Lemma 9.6. *The algebra $H^A_\bullet(Z)$ contains $H_\bullet(Z)$ as its subalgebra. In particular, we have $\mathbb{C}[W] \subset H^A_\bullet(Z)$ as subalgebras. Moreover, the center of $H^A_\bullet(Z)$ contains $H^\bullet(BA)[(\triangle F)_A] \subset H^A_\bullet(Z)$.*

Proof. In the Leray spectral sequence

$$H^i(BA) \otimes H_j(Z) \Rightarrow H^A_{-i+j}(Z),$$

we have $H^{odd}(BA) = 0$ and $H_{odd}(Z) = 0$ (since Z is paved by affine spaces). It follows that this spectral sequence degenerates at the level of E_2-terms. Moreover, the image of the natural map $\imath : H_j(Z) \hookrightarrow H^A_j(Z)$ represents cycles which are locally constant fibration over the base BA. It follows that the map $\imath$ is an embedding of convolution algebras.
Multiplying $H^\bullet(BA)$ is an operation along the base BA, which commutes with the convolution operation (along the fibers of f^A_Z). It follows that $H^\bullet(BA) \to H^\bullet(BA)[(\triangle F)_A] \subset H^A_\bullet(Z)$ is central subalgebra as desired. □

We return to the proof of Proposition 9.2.

By the Thomason localization theorem (see e.g. [CG97] §8.2), we have an isomorphism

$$R(A)_{\underline{a}} \otimes_{R(A)} K^A(Z^{\underline{a}}) \cong R(A)_{\underline{a}} \otimes_{R(A)} K^A(Z)$$

as algebras. For each of $\mathcal{X} = Z$, or $Z^{\underline{a}}$, we have an embedding

$$\imath : K^A(\mathcal{X}) \hookrightarrow \varprojlim_m K^A(ET_m \times \mathcal{X}) \cong \varprojlim_m K(A \backslash (ET_m \times \mathcal{X}))$$

obtained by pulling back an A-equivariant vector bundle on an irreducible component of $\mathcal{X}$ to each $ET_m \times \mathcal{X}$ by the second projection. The latter inverse limits are formed by the pullbacks via the closed embeddings coming from (9.3). Here the last inverse limit has a natural topology whose open sets are formed by the formal sum of vector bundles which are trivial on $K(A \backslash (ET_m \times \mathcal{X}))$ for some fixed choice of m. By construction, the image of $\imath$ must be dense open with respect to the topology on the RHS.

We regard the RHS as a substitute of $K(\mathcal{X}_A)$. Let $\mathbb{C}[[\mathfrak{a}]]_a$ and $\mathbb{C}[\mathfrak{a}]_a$ be the formal power series ring and the localized ring of $\mathbb{C}[\mathfrak{a}]$ along $\log \underline{a}$, respectively. The Chern character map relative to BA gives an isomorphism

$$\mathbb{C}[[\mathfrak{a}]]_a \otimes_{\mathbb{C}[\mathfrak{a}]} H^A_\bullet(Z^{\underline{a}}) \cong \mathbb{C}[[\mathfrak{a}]]_a \otimes_{\mathbb{C}[\mathfrak{a}]} H^A_\bullet(Z).$$

By restricting this to the sum of vectors of finitely many degrees, we obtain

$$\mathbb{C}[\mathfrak{a}]_a \otimes_{\mathbb{C}[\mathfrak{a}]} H^A_\bullet(Z^{\underline{a}}) \cong \mathbb{C}[\mathfrak{a}]_a \otimes_{\mathbb{C}[\mathfrak{a}]} H^A_\bullet(Z). \tag{9.4}$$

Since localization along $\mathbb{C}[\mathfrak{a}]_a$ commutes with the quotient by its unique maximal ideal, we deduce the first assertion.

The isomorphism (9.4) is an algebra isomorphism. This implies that $1 \in \mathbb{C}[W]$ goes to $1 \in H_\bullet(Z^{\underline{a}})$. It follows that each of s_i goes to a non-zero element of $H_\bullet(Z^{\underline{a}})$ with its square equal to 1. By construction, there exists $f_i \in \mathbb{C}(\mathfrak{a})$ $(i = 1, \ldots n)$ such that $1, f_1 s_1, \ldots f_n s_n \in H^A_\bullet(Z^{\underline{a}})$ define linearly independent vectors in $H_\bullet(Z^{\underline{a}})$. It follows that $f_i^2 \in \mathbb{C}[\mathfrak{a}]$. This forces $f_i \in \mathbb{C}[\mathfrak{a}]$ (since a polynomial ring is integrally closed), which implies that the images of $1, s_1, \ldots, s_n \in H_\bullet(Z^{\underline{a}})$ are linearly independent. This verifies the second assertion.

The vector space $H^A_\bullet(\mu^{-1}(X))$ admits an action of $H^A_\bullet(Z)$. By the Leray spectral sequence, we have

$$H^\bullet(BA) \otimes H_\bullet(\mu^{-1}(X)) \Rightarrow H^A_\bullet(\mu^{-1}(X)). \tag{9.5}$$

By Theorem 6.2 and Corollary 9.5, we know that $H_{odd}(\mu^{-1}(X)) = 0 = H^{odd}(BA)$. It follows that (9.5) is E_2-degenerate, which proves the last part of Proposition 9.2.

10 Main Theorems

We retain the setting of §2.

Theorem 10.1 (Deligne-Langlands type classification)**.** *Let $a \in \boldsymbol{G}$ be a finite pre-admissible element. Then, $\mathfrak{R}_a$ is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.*

Proof. By definition, each element of $\mathfrak{R}_a$ corresponds to at least one isomorphism class of $\mathbb{H}_a$-modules. Since a is finite, each irreducible direct summand of $(\mu^a_+)_* \mathbb{C}_{F^a_+}$ is the minimal extension of a local system (up to degree shift) from a $\boldsymbol{G}(a)$-orbit $\mathbb{O}$. By Theorem 4.10, a $\boldsymbol{G}(a)$-equivariant local system on $\mathbb{O}$ is a constant sheaf. As a result, every element of $\mathfrak{R}_a$ corresponds to at most one irreducible module as desired. □

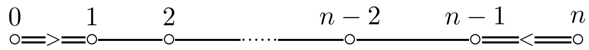
Theorem 10.2 (Effective Deligne-Langlands type classification)**.** *Let $a \in \boldsymbol{G}$ be an admissible element. Then, the set Λ_a is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.*

Proof. The proof is given at the end of this section. □

As in Remark 2.2, the quotient $\mathbb{H}/(\mathbf{q}_0 + \mathbf{q}_1)$ is isomorphic to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two parameters. Hence, we have

Corollary 10.3 (Effective Deligne-Langlands type classification for type B)**.** *Let $a = (s, q_0, -q_0, q_2) \in \boldsymbol{G}$ be a pre-admissible element such that $-q_0^2 \neq q_2^{\pm m}$ holds for every $0 \leq m < n$. Then, the set Λ_a is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.* □

Remark 10.4. The Dynkin diagram of type $C_n^{(1)}$ is written as:



This Dynkin diagram has a unique non-trivial involution φ. We define t_0, t_1, t_n to be

$$t_1^2 = \mathbf{q}_2, t_n^2 = -\mathbf{q}_0\mathbf{q}_1, t_n(t_0 - t_0^{-1}) = \mathbf{q}_0 + \mathbf{q}_1 \quad \text{(c.f. Remark 2.2 1)).}$$

Let $T_0, \ldots, T_n$ be the Iwahori-Matsumoto generators of $\mathbb{H}$ (c.f. [Mc03, Lu03]). Their Hecke relations read

$$(T_0 + 1)(T_0 - t_0^2) = (T_i + 1)(T_i - t_1^2) = (T_n + 1)(T_n - t_n^2) = 0,$$

where $1 \le i < n$. The natural map $\varphi(T_i) = T_{n-i}$ $(0 \le i \le n)$ extends to an algebra map $\varphi : \mathbb{H} \to \mathbb{H}'$, where $\mathbb{H}'$ is the Hecke algebra of type $C_n^{(1)}$ with parameters t_n, t_1, t_0. We have $t_n = \pm\sqrt{-\mathbf{q}_0\mathbf{q}_1}$ and $t_0 = \pm\sqrt{-\mathbf{q}_0/\mathbf{q}_1}$ or $\pm\sqrt{-\mathbf{q}_1/\mathbf{q}_0}$. In particular, φ changes the parameters as $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) \mapsto (\mathbf{q}_0, \mathbf{q}_1^{-1}, \mathbf{q}_2)$ or $(\mathbf{q}_0^{-1}, \mathbf{q}_1, \mathbf{q}_2)$. Therefore, the representation theory of $\mathbb{H}_a$ ($a = (s, \vec{q})$) is unchanged if we replace q_0 with q_0^{-1}, or q_1 with q_1^{-1}.

The rest of this section is devoted to the proof of Theorem 10.3. In the course of the proof, we use:

Proposition 10.5. *Let $a = (s, \vec{q}) \in \boldsymbol{T}$ be an admissible element. Let $\mathcal{O} \subset \mathfrak{N}$ be a G-orbit. For any two distinct $G(s)$-orbits $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O} \cap \mathfrak{N}_+^a$, we have*

$$\overline{\mathcal{O}_1} \cap \mathcal{O}_2 = \emptyset.$$

Proof. By Proposition 4.8 and Lemma 1.18, we deduce that the scalar multiplication of a normal form of $\mathfrak{N}$ is achieved by the action of T. It follows that each $G(s)$-orbit of $\mathfrak{N}^a$ is a $Z_{\boldsymbol{G}}(a)$-orbit. Let $X \in \mathcal{O}_1$. Let G_X be the stabilizer of X in $\boldsymbol{G}$. Assume that $\mathcal{O}_2 \cap \overline{\mathcal{O}_1} \neq \emptyset$ to deduce contradiction. Since $\mathcal{O}_2$ is a $Z_{\boldsymbol{G}}(a)$-orbit, we have $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$. Fix $X' \in \mathcal{O}_2$. Consider an open neighborhood $\mathcal{U}$ of 1 in G (as complex analytic manifolds). Then, $\mathcal{U}X' \in \mathcal{O}$ is an open neighborhood of X'. It follows that $\mathcal{U}X' \cap \mathcal{O}_1 \neq \emptyset$. We put $\mathfrak{g}_{a,X'} := \mathrm{Lie} G_{X'} + \mathrm{Lie} Z_{\boldsymbol{G}}(a)$. We have

$$N_{\mathcal{O}_2/\mathcal{O},X'} = \mathfrak{g}/\mathfrak{g}_{a,X'}.$$

Every non-zero vectors of $N_{\mathcal{O}_2/\mathcal{O},X'}$ is expressed as a linear combination of eigenvectors with respect to the a-action. These a-eigenvectors can be taken to have non-zero weights and does not contained in $G_{X'}$. It follows that

$$\mathcal{U}X' \cap \mathcal{O}_1 \not\subset \mathbb{V}^a,$$

which is contradiction (for an arbitrary sufficiently small $\mathcal{U}$). Hence, we have necessarily $\mathcal{O}_2 \cap \overline{\mathcal{O}_1} = \emptyset$ as desired. □

Proof of Theorem 10.2. By taking G-conjugate if necessary, we assume $a \in \boldsymbol{T}$. By Corollary 3.10, it suffices to prove Theorem 10.2 when $\mathcal{C}_a$ consists of a unique clan $[1, n]$. By Corollary 4.9, we can further assume $V_1^{(s,q_1)} = \{0\}$ by swapping the roles of q_0 and q_1 if necessary. By Theorem 8.5 (c.f. Theorem 1.20), an admissible parameter (a, X) is regular if there exists a simple $\mathbb{H}_a$-constituent of $M_{(a,X)}$ which does not appear in any $M_{(a,X')}$ such that $\overline{G(s)X} \subsetneq \overline{G(s)X'}$.

We apply Proposition 9.1 (if necessary) to modify a so that the assumption of Proposition 9.2 is fillfulled. By Proposition 9.2, each $M_{(a,X)}$ has a W-module structure given by the restriction of the $\mathbb{H}_a$-module structure. Moreover, the simple W-module L_X corresponding to the G-orbit $GX \subset \mathfrak{N}$ (by the exotic Springer correspondence) appears in $M_{(a,X)}$. By Proposition 10.5, we have $GX \neq GX'$ for every $X' \in \mathfrak{N}^a$ such that $\overline{G(s)X} \subsetneq \overline{G(s)X'}$. By Corollary 9.3 and Corollary 8.6, $M_{(a,X')}$ does not contain L_X as W-modules. Hence, the simple $\mathbb{H}_a$-constituent of $M_{(a,X)}$ which contains L_X as W-type does not occur in any $M_{(a,X')}$ such that $\overline{G(s)X} \subsetneq \overline{G(s)X'}$ as required. □

11 Consequences

In this section, we present some of the consequences of our results. We retain the setting of the previous section.

Definition 11.1. Let $\nu = (a, X)$ be an admissible parameter. Let L_ν be the simple module of $\mathbb{H}$ corresponding to ν. Let $IC(\nu)$ be the corresponding $\boldsymbol{G}(a)$-equivariant simple perverse sheaf on $\mathfrak{N}^a_+$. (c.f. §1.4) We denote by P_ν the projective cover of L_ν as $\mathbb{H}_a$-modules. (It exists since $\mathbb{H}_a$ is finite dimensional.)

Let K be a $\mathbb{H}$-module and let L be a simple $\mathbb{H}$-module. We denote by $[K : L]$ the multiplicity of L in K.

Applying [CG97] 8.6.23 to our situation, we obtain:

Theorem 11.2 (The multiplicity formula of standard modules)**.** *Let $\nu = (a, X)$, $\nu' = (a, X')$ be admissible parameters. We have:*

$$[M_\nu : L_{\nu'}] = \sum_k \dim H^k(i^!_X IC(\nu')) \text{ and } [M^\nu : L_{\nu'}] = \sum_k \dim H^k(i^*_X IC(\nu')),$$

where $i_X : \{X\} \hookrightarrow \mathfrak{N}^a_+$ is an inclusion. □

The following result is a variant of the Lusztig-Ginzburg character formula of standard modules in our setting.

Theorem 11.3 (The character formula of standard modules)**.** *Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter. Let $\mathfrak{B}_\nu$ be the set of connected components of $\mathcal{E}^a_X$. For each $\mathcal{B} \in \mathfrak{B}_\nu$, we define a linear form $\langle \bullet, s\rangle_{\mathcal{B}}$ as a composition map*

$$\begin{tikzcd} \langle \bullet, s\rangle_{\mathcal{B}} : R(T) \arrow[r, "\cong"] & R(gBg^{-1}) \arrow[r, "\mathrm{ev}_s"] & \mathbb{C} \\ R^+ \arrow[u, hook] \arrow[r] & \{\text{weights of } gBg^{-1}\} \arrow[u, hook] & \end{tikzcd}$$

by some $g \in G$ such that $gB \in \mathcal{B}$. Then, $\langle \bullet, s\rangle_{\mathcal{B}}$ is independent of the choice of g and the restriction of M_ν to $R(T)$ is given as

$$\mathrm{Tr}(e^\lambda; M_\nu) := \sum_{\mathcal{B} \in \mathfrak{B}_\nu} \langle \lambda, s\rangle_{\mathcal{B}} \sum_{j \geq 0} \dim H_{2j}(\mathcal{B}, \mathbb{C}).$$

Proof. Taking account into Theorem 4.10, the proof is exactly the same as in [CG97] §8.2. □

Definition 11.4. Let $a = (s, \vec{q}) \in \boldsymbol{T}$ be an admissible element. We form three $|\Lambda_a| \times |\Lambda_a|$-matrices

$$[P : L]^a_{\nu,\nu'} := [P_\nu, L_{\nu'}], D^a_{\nu,\nu'} := \delta_{\nu,\nu'} \chi_c(\nu), \text{ and } IC^a_{\nu,\nu'} := [M^\nu, L_{\nu'}],$$

where $\chi_c(\nu) := \sum_{i \ge 0} (-1)^i \dim H^i(\boldsymbol{G}(a)X, \mathbb{C})$ $(\nu = (a, X))$.

The following result is a special case of the Ginzburg theory [CG97] Theorem 8.7.5 applied to our particular setting:

Theorem 11.5 (The multiplicity formula of projective modules)**.** *Keep the setting of Definition 11.4. We have*

$$[P : L]^a = IC^a \cdot D^a \cdot {}^t IC^a,$$

where ^{t} denotes the transposition of matrices. □

Index of notation

(Sorted by the order of appearance)

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